Basic Examination: Functional Analysis   September 2020

- This test is **closed book**: no notes or other aids are permitted.
- You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).
- You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. Let $X$ be the Banach space of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous and bounded, with norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$. For each $t \in \mathbb{R}$ define the operator $U(t) \in \mathcal{L}(X)$ (the space of bounded linear operators on $X$) by
   $$(U(t)f)(x) = f(x + t).$$

   (a) Prove that $X$ is not separable.
   (b) Is the map $t \mapsto U(t)$ continuous from $\mathbb{R}$ to $\mathcal{L}(X)$? Prove or disprove.
   (c) Let $Y$ be the set of $f \in X$ such that the strong limit
   $$\lim_{t \to 0} \frac{U(t)f - f}{t}$$
   exists in $X$. Show $Y$ is not dense in $X$.

2. Let $S$ be a closed convex set in a Hilbert space $H$. For each $x \in H$ let $g(x) \in S$ be the unique element in $S$ such that
   $$\|x - g(x)\| = \inf_{z \in S} \|x - z\|.$$  
   Show that $y = g(x)$ if and only if $y \in S$ and $(x - y, z - y) \leq 0$ for all $z \in S$.

3. In the Hilbert space $H = L^2([0,1])$, define the operator $T$ by
   $$(Tf)(x) = \int_0^1 \min(x,y)f(y) \, dy$$

   (i) Show that $T$ is compact, and (ii) find its spectrum.

4. Let $X$ be a real Banach space. Let $p : X \to [0, \infty)$ be such that $p(x+y) \leq p(x) + p(y)$ and $p(tx) = tp(x)$, for all $x,y \in X$ and $t \geq 0$. Prove that the set $B = \{x \in X : p(x) < 1\}$ is convex. Is it necessarily the case that $B$ contains a neighborhood of 0? Prove your answer.

5. Let $X$ and $Y$ be real Banach spaces with duals $X^*$ and $Y^*$ respectively, and let $\mathcal{L}(X,Y)$ denote the space of bounded linear operators $T : X \to Y$.

   (a) Give explicitly a neighborhood base at 0 for $\sigma(X^*,X)$, the weak-* topology of $X^*$.
   (b) Given any $T \in \mathcal{L}(X,Y)$, prove that its adjoint $T'$ is a continuous map from the topological vector space $Y^*$ equipped with the topology $\sigma(Y^*,Y)$, to $X^*$ equipped with the topology $\sigma(X^*,X)$.
   (c) Show that if $f$ is a linear functional on $X^*$ which is weak-* continuous, then there is some $x \in X$ such that $f(\ell) = \ell(x)$ for all $\ell \in X^*$. (Hint: show codim ker $f < \infty$.)