Basic Examination: Functional Analysis September 2020

- This test is **closed book**: no notes or other aids are permitted.
- You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).
- You may use without proof *standard* results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **clearly** state the result you are using.

1. Let X be the Banach space of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous and bounded, with norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. For each $t \in \mathbb{R}$ define the operator $U(t) \in \mathcal{L}(X)$ (the space of bounded linear operators on X) by

$$(U(t)f)(x) = f(x+t).$$

(a) Prove that X is not separable.

(b) Is the map $t \mapsto U(t)$ continuous from $\mathbb{R}$ to $\mathcal{L}(X)$? Prove or disprove.

(c) Let Y be the set of $f \in X$ such that the strong limit

$$\lim_{t\to 0} \frac{U(t)f - f}{t}$$

exists in X. Show Y is not dense in X.

2. Let S be a closed convex set in a Hilbert space H. For each $x \in H$ let $g(x) \in S$ be the unique element in S such that

$$\|x - g(x)\| = \inf_{z\in S} \|x - z\|.$$

Show that $y = g(x)$ if and only if $y \in S$ and $(x - y, z - y) \le 0 \quad$ for all $z \in S$.

3. In the Hilbert space $H = L^2([0,1])$, define the operator T by

$$(Tf)(x) = \int_0^1 \min(x,y) f(y)\, dy$$

(i) Show that T is compact, and (ii) find its spectrum.

4. Let X be a real Banach space. Let $p : X \to [0,\infty)$ be such that $p(x+y) \le p(x) + p(y)$ and $p(tx) = tp(x)$, for all $x, y \in X$ and $t \ge 0$. Prove that the set $B = \{x \in X : p(x) < 1\}$ is convex. Is it necessarily the case that B contains a neighborhood of 0? Prove your answer.

5. Let X and Y be real Banach spaces with duals X^* and Y^* respectively, and let $\mathcal{L}(X,Y)$ denote the space of bounded linear operators $T : X \to Y$.

(a) Give explicitly a neighborhood base at 0 for $\sigma(X^*, X)$, the weak-$*$ topology of X^*.

(b) Given any $T \in \mathcal{L}(X,Y)$, prove that its adjoint T' is a continuous map from the topological vector space Y^* equipped with the topology $\sigma(Y^*, Y)$, to X^* equipped with the topology $\sigma(X^*, X)$.

(c) Show that if f is a linear functional on X^* which is weak-$*$ continuous, then there is some $x \in X$ such that $f(\ell) = \ell(x)$ for all $\ell \in X^*$. (Hint: show $\operatorname{codim} \ker f < \infty$.)