2020 January Basic Exam of Functional Analysis

January 16th 4:30pm-7:30pm

Closed book and closed notes.

(1) (15 points) Given $\alpha \in (0,1)$, consider the space $C^\alpha([0,1])$ consisting of all functions with the norm

$$\|f\|_\alpha = \sup_{x\in[0,1]} |f(x)| + \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

(i) show it is a Banach space; (ii) show that any bounded sequence $\{f_n\}$ in $C^\alpha([0,1])$ has a subsequence $\{f_{n_j}\}$ that converges uniformly in $[0,1]$.

(2) (15 points) Let H be a Hilbert space and $M \subset H$ be a subspace. Let $f : H \to \mathbb{C}$ be a linear functional on M with bound $C = \sup_{x\in M,\|x\|=1} |f(x)|$. Prove that there is a *unique* extension of f to a continuous linear functional on H with the same bound.

(3) (20 points) Let $T : H \to H$ be a self-adjoint bounded operator on the Hilbert space H, and assume T is bounded from below in the sense that $\|Tx\| \geq c\|x\|$ for some $c > 0$ and all $x \in H$. Show that for any $y \in H$, there is a unique solution to $Tx = y$.

(4) (20 points) Let X be a Banach space, Y and Z are closed subspaces of X that complement each other: $X = Y \bigoplus Z$, in the sense that every x in X can be decomposed uniquely $x = y + z$ with $y \in Y, z \in Z$. Denote $y = P_Y x$ and $z = P_Z x$. Show that (i) P_Y and P_Z are linear maps; (ii) $P_Y^2 = P_Y, P_Z^2 = P_Z, P_Y P_Z = 0$; (iii) P_Y and P_Z are continuous.

(5) (15 points) Consider the operator $A : \ell_2 \to \ell_2$ defined as $A(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots)$, where $\lambda_n \in \mathbb{C}$. Show that A is compact iff $\lambda_n \to 0$ as $n \to \infty$.

(6) (15 points) Consider the space $\ell_2(\mathbb{Z})$, i.e., the set of $(\ldots, x_{-1}, x_0, x_1, \ldots)$ such that $\sum_{k\in\mathbb{Z}} |x_k|^2 < \infty$. Define the discrete Laplacian operator on $\ell_2(\mathbb{Z})$ by $(\Delta x)_k = x_{k-1} + x_{k+1} - 2x_k$. Find and classify the spectrum of Δ.