## 2019 Basic Exam: Functional Analysis

*Sept 5th 4:30pm-7:30pm*

*Closed book and closed notes.*

(1) Let $H$ be a Hilbert space. Assuming $\{T_n\}$ is a sequence of bounded linear operators from $H$ to $H$ such that for all $x, y \in H$ we have $\langle T_n x, y\rangle$ converges as $n \to \infty$. Show that there exists a unique bounded linear operator $T$ such that $\langle T_n x, y\rangle \to \langle Tx, y\rangle$.

(2) Let $X, Y$ be Banach spaces, $T : X \to Y$ is a bounded linear map. Show that the following are equivalent:
(i) there exists $c > 0$ such that $\|x\|_X \le c\|Tx\|_Y$ for all $x \in X$;
(ii) $T$ has a closed range and the only solution to $Tx = 0$ is $x = 0$.

(3) Define $T : \ell_1 \to \ell_1$ by $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$, find and classify the spectrum of $T$.

(4) Consider the operator $K$ defined on $L^2[0,1]$ as

$$Kf(x) = \int_x^1 \left( \int_0^y f(z)dz \right) dy$$

(i) Show that $K$ is self-adjoint; (ii) show that $K$ is compact; (iii) find the spectrum of $K$.

(5) Let $H$ be a Hilbert space. Consider a bounded linear operator $K : H \to H$, show that $K$ is compact if and only if $K$ maps weakly convergent sequences into strongly convergent sequences.

(6) (i) State the Hahn-Banach theorem; (ii) show $\ell_\infty = \ell_1^*$; (iii) use Hahn-Banach theorem to show that $\ell_1$ is not reflexive.