Carnegie Mellon University **Department of Mathematical Sciences**

Functional Analysis Basic Exam – Fall 2018

Closed-book and closed-notes, three hours

Answer questions 1 and 2 plus any other six questions (8 total). If you answer all of the last 7 questions, I will use your best 6 for computing your grade. (You have nothing to lose by attempting all questions.) Be sure to give complete and clear explanations in your answers to questions 3 thru 9.

1. (20 pts) Let l^2 denote the set of all real-valued square-summable sequences, equipped with the usual norm

$$\|x\| = \left(\sum_{k=1}^{\infty} (x_k)^2 \right)^{\frac{1}{2}} \quad \text{for all } x \in l^2.$$

Define $T : l^2 \to l^2$ by

$$(Tx)_k = \begin{cases} 0 \text{ if } k \text{ is odd} \\ x_{\frac{k}{2}} \text{ if } k \text{ is even} \end{cases} \quad \text{for all } x \in l^2,$$

i.e.

$$Tx = (0, x_1, 0, x_2, 0, x_3, \cdots).$$

You may take it for granted that T is a bounded linear operator from l^2 to l^2.

(a) Is T injective? (Give a brief explanation.)

(b) Is $\mathcal{R}(T)$ dense in l^2? (Give a brief explanation.)

(c) Assuming that we identify the dual space $(l^2)^*$ with l^2 in the usual way, find an expression for the adjoint operator T^*.

(d) Is T compact? Explain.

(e) Give an example of a bounded linear mapping $L : l^2 \to l^2$ such that $TL \neq LT$.

2. (20 pts)

(a) State the Hahn-Banach Theorem for linear spaces in geometric form (i.e., state a result about separation of convex sets).

(b) State the Banach-Steinhaus Theorem (also called the Principle of Uniform Boundedness).

(c) State the Closed Graph Theorem.

(d) State the Riesz-Representation Theorem for Hilbert spaces.

3. (10 pts) Let X and Y be real or complex normed linear spaces, $T : X \to Y$ be a linear mapping and $x_0 \in X$ be given. Show that if T is continuous at x_0 then T is uniformly continuous (on X).

4. (10 pts) Let X be a real or complex normed linear space and let $B \subset X$. (Let's assume that B is nonempty.) Show that B is bounded if and only if

$$\forall x^* \in X^*, \quad \sup\{|x^*(x)| : x \in B\} < \infty.$$

5. (10 pts) Let X be a real or complex topological vector space, $f : X \to \mathbb{K}$ be a nontrivial continuous linear functional. Prove or disprove: If $A \subset X$ is open and convex then $f[A] = \{f(x) : x \in A\}$ is open and convex in $\mathbb{K}$. (If you can't answer the question when X is a TVS, I will give 5 points for the correct answer when X is a Banach space.)

6. (10 pts) Let X be a real or complex Banach space and let $x_0 \in X$ be given. Put

$$V = \{\alpha x_0 : \alpha \in \mathbb{K}\}.$$

Show that there is a closed subspace W of X with the following property: For every $x \in X$ there is exactly one pair $(v, w) \in V \times W$ such that $x = v + w$. (Please prove this result "from scratch". Do not simply quote a theorem stating that finite-dimensional subspaces are complemented.)

7. (10 pts) Let X be a real or complex Hilbert space with inner product $(\cdot, \cdot)$ and $\{x_n\}_{n=1}^{\infty}$ be an orthonormal sequence of elements of X, i.e.

$$(x_n, x_m) = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}.$$

Show that $x_n \rightharpoonup 0$ (weakly) as $n \to \infty$.

8. (10 pts) Let X be complex Hilbert space with inner product $(\cdot, \cdot)$ and let $T : X \to X$ be a linear mapping satisfying

$$(Tx, y) = i(x, Ty) \text{ for all } x, y \in X.$$

Here $i^2 = -1$. Show that T is continuous.

9. (10 pts) Let X be a reflexive real Banach space and $x^* \in X^*$ be given. Define $g : X \to \mathbb{R}$ by

$$g(x) = \|x\|^2 - x^*(x) \text{ for all } x \in X.$$

Show that there exists $x_0 \in X$ such that

$$g(x) \geq g(x_0) \text{ for all } x \in X.$$

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Functional Analysis Basic Exam – Explanation of Symbols

$\mathcal{N}(T)$ – the null space of T

$\mathcal{R}(T)$ – the range of T

$\mathcal{L}(X;Y)$ – the set of all bounded linear operators from X to Y

X^* – the topological dual space of a linear space X

T^* – the adjoint of a bounded linear operator

T^{-1} – the inverse of a bijective linear operator

$\mathbb{N} = \{1, 2, 3, \cdots\}$ the set of all strictly positive integers

$\mathbb{K}$ – a field that is either $\mathbb{R}$ or $\mathbb{C}$

$x_n \to x$: x_n converges strongly to x

$x_n \rightharpoonup x$: x_n converges weakly to x

$\operatorname{int}(S)$ – the interior of a set S

$\operatorname{cl}(S)$ – the closure of a set S

$f[S] = \{f(x) : x \in S\}$

Remark: My definition of topological vector space includes the Hausdorff property.