Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION: FUNCTIONAL ANALYSIS
September 7, 2017, 4:30pm-7:30pm

1. (i) State the Closed Graph Theorem.

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces.
(ii) Let $T : X \to Y$ be a linear and continuous operator such that Range(T) is closed. Prove that there exists $C > 0$ such that for every $y \in T(X)$ there exists $x \in X$ such that

$$y = T(x) \quad \text{and} \quad ||x||_X \leq C||y||_Y.$$

(iii) Let $T : X \to Y$ be a linear operator such that for every sequence $\{x_n\} \subset X$

$$||x_n|| \to 0 \Rightarrow T(x_n) \rightharpoonup 0 \quad \text{in } \sigma(Y, Y').$$

Prove that T is continuous, i.e., $T \in \mathcal{L}(X; Y)$.

2. (i) Prove that if $(X, ||\cdot||)$ is a normed space over $\mathbb{R}$ such that X' is separable, then X is also separable.

(ii) Let $(X, ||\cdot||)$ be an infinite dimensional Banach space. Prove that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that

$$||x_n|| = 1 \quad \text{for all}\, n \in \mathbb{N}, \text{and}\ x_n \rightharpoonup 0.$$

3. Let X and Y be a normed spaces, $X \neq \{0\}$. Prove that Y is a Banach space if and only if the normed space $\mathcal{L}(X; Y)$ is a Banach space.

4. (i) Give the definition of a compact operator between normed spaces.

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed spaces over $\mathbb{R}$.
(ii) Assume that X is a reflexive Banach space, Y is a Banach space, and let $T \in \mathcal{L}(X; Y)$ be such that for every sequence $\{x_n\} \subset X$

$$x_n \rightharpoonup x \quad \text{in } \sigma(X; X') \Rightarrow T(x_n) \to T(x).$$

Prove that T is compact.

(iii) Let $T : X \to Y$ be a linear compact operator. Prove that $T^\star$ is also compact.

5. Let $(H, (\cdot,\cdot))$ be a Hilbert space and let $T : H \to H$ be a linear, continuous operator such that

$$(Tx, x) \geq 0 \quad \text{for all } x \in H.$$

(OVER)

(i) Prove that $\text{Ker}(T) = (\text{Range}(T))^{\perp}$.

(ii) Prove that $\mathbb{I} + tT$ is bijective for all $t > 0$.

Fix $x \in H$.

(iii) Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $t_n \to +\infty$ and let x_n satisfy

$$x_n + t_n T x_n = x$$

for all $n \in \mathbb{N}$. Prove that $||x_n|| \leq ||x||$ and that (up to a subsequence) $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to some $\tilde{x} \in \text{Ker}(T)$.

(iv) Prove that $\tilde{x} = \text{Proj}_{\text{Ker}(T)} x$.

(v) Prove that

$$\lim_{t\to+\infty} (\mathbb{I} + tT)^{-1} x = \text{Proj}_{\text{Ker}(T)} x.$$