Carnegie Mellon University **Department of Mathematical Sciences**

Functional Analysis Basic Exam – Fall 2016

Closed-book and closed-notes, three hours

Answer questions 1 and 2 plus any other four questions (6 total). If you answer more than 4 of the last 6 questions, be sure to indicate which 4 you would like to have graded.

1. Let l^2 denote the set of all real-valued square-summable sequences, equipped with the usual norm

$$\|x\| = \left(\sum_{k=1}^{\infty}(x_k)^2\right)^{\frac{1}{2}} \quad \text{for all } x \in l^2.$$

Define $T : l^2 \to l^2$ by

$$(Tx)_k = \frac{x_{k+2}}{k+2} \quad \text{for all } x \in l^2,\ \ k = 1, 2, 3, \cdots .$$

You may take it for granted that T is a bounded linear operator from l^2 to l^2.

   (a) Find $\|T\|$.

   (b) Find $\mathcal{N}(T)$ and $\mathcal{R}(T)$, the null space and range of T.

   (c) Assuming that we identify the dual space $(l^2)^*$ with l^2 in the usual way, find an expression for the adjoint operator T^*.

   (d) Is T compact? Explain.

2. (a) State the Hahn-Banach Theorem (in extension form) for linear spaces.

   (b) State the Banach-Steinhaus Theorem (also called the Principle of Uniform Boundedness).

   (c) State the Open-Mapping Theorem.

   (d) Give an example of a Banach space X, a normed linear space Y and a continuous linear bijection $T : X \to Y$ such that T^{-1} is not continuous.

3. (a) Give an example of a Banach space Y, a bounded linear operator $L \in \mathcal{L}(Y;Y)$ and a sequence $\{L_n\}_{n=1}^{\infty}$ in $\mathcal{L}(Y;Y)$ such that

$$\forall y \in Y, \quad L_n y \to Ly \ \text{ as } n \to \infty,$$

   but $\|L_n - L\| \not\to 0$ as $n \to \infty$.

(b) Let X be a Banach space, $T \in \mathcal{L}(X;X)$ and a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{L}(X;X)$ be given. Let $K : X \to X$ be a compact linear operator. Assume that

$$\forall x \in X, \quad T_n x \to Tx \quad \text{as } n \to \infty.$$

Show that $\|T_n K - TK\| \to 0$ as $n \to \infty$.

4. Let X be a Banach space over $\mathbb{K}$, and $T \in \mathcal{L}(X;X)$ be given. Recall that a scalar $\lambda \in \mathbb{K}$ is called a *generalized eigenvalue* for T provided that there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$\|x_n\| = 1 \ \text{ for all } n \in \mathbb{N} \quad \text{and} \quad (\lambda I - T)x_n \to 0 \ \text{ as } n \to \infty.$$

(a) Let $T : X \to X$ be a compact linear operator and $\lambda \in \mathbb{K}$ with $\lambda \neq 0$ be given. Show that if λ is a generalized eigenvalue for T then λ is an eigenvalue for T.

(b) Give an example of a Banach space Y and a compact linear mapping $K : Y \to Y$ such that 0 is a generalized eigenvalue for K, but not an eigenvalue for K.

5. (a) Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Show that T is bijective if and only if there exist constants $c, k > 0$ such that

$$\|Tx\| \geq c\|x\| \ \text{ for all } x \in X \quad \text{and} \quad \|T^* y^*\| \geq k\|y^*\| \ \text{ for all } y^* \in Y^*.$$

(Here X^*, Y^* are the dual space of X, Y and T^* is the adjoint of T.) You may make use of basic results about operators and their adjoints.

(b) If X and Y are normed linear spaces (not necessarily complete), does the result of part (a) hold? Prove or give a counterexample.

6. Let X and Y be Banach spaces over $\mathbb{K}$ and let $B : X \times Y \to \mathbb{K}$ be a bilinear mapping, i.e.

$$B(\alpha x_1 + \beta x_2, y) = \alpha B(x_1, y) + \beta B(x_2, y) \ \text{ for all } \alpha, \beta \in \mathbb{K},\ x_1, x_2 \in X,\ y \in Y,$$

$$B(x, \alpha y_1 + \beta y_2) = \alpha B(x, y_1) + \beta B(x, y_2) \ \text{ for all } \alpha, \beta \in \mathbb{K},\ x \in X,\ y_1, y_2 \in Y.$$

Assume that

$$\forall y \in Y,\ B(\cdot, y) : X \to \mathbb{K} \text{ is continuous}$$

and

$$\forall x \in X,\ B(x, \cdot) : Y \to \mathbb{K} \text{ is continuous.}$$

Show that $B : X \times Y \to \mathbb{K}$ is (jointly) continuous.

7. Let X be a topological vector space and let C be a convex subset of X.

(a) Show that $\text{int}(C)$ is convex.

(b) Show that $\text{cl}(C)$ is convex.

(c) If C is closed and $f : X \to \mathbb{K}$ is linear and continuous, does it follow that $f[C]$ is closed? Prove or give a counterexample.

8. Let X be a real or complex Hilbert space with inner product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$.

   (a) Let $y \in X$ and a sequence $\{y_n\}_{n=1}^{\infty}$ in X be given and assume that $y_n \rightharpoonup y$ (weakly) as $n \to \infty$ and that $\|y_n\| \to \|y\|$ as $n \to \infty$. Show that $y_n \to y$ (strongly) as $n \to \infty$.

   (b) Let $T \in \mathcal{L}(X;X)$ and a sequence $\{x_n\}_{n=1}^{\infty}$ be given. Assume that T is self-adjoint and that $\|T\| = 1$. Assume further that $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$ and that $\|Tx_n\| \to 1$ as $n \to \infty$. Show that

$$\|T^2 x_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$