Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION: FUNCTIONAL ANALYSIS
September 3, 2015, 4:30pm-6:30pm

Solve all five problems

1. (i) Give the definition of a compact operator between normed spaces.

(ii) Let X and Y be normed spaces and let $T \in \mathcal{L}(X;Y)$ be a compact operator.

(a) Prove that range$T$ is separable.

(b) Prove that the adjoint $T^\star$ is also a compact operator.

(c) Suppose that $X = Y$. What can you say (without proofs) about the spectrum of T, $\sigma(T)$, and the point spectrum, $\sigma_p(T)$? In particular:

(c)-1 If X is Banach, can there be a sequence $\{\lambda_n\} \subset \mathbb{C}$ in $\sigma(T)$ such that $|\lambda_n| \to \infty$? Justify.

(c)-2 If X is an infitine dimensional Banach space, can $0 \in \sigma(T)$? Is there an interval of real numbers in $\sigma(T)$? Justify your responses.

2. Let X be a Banach space and let Y be a closed subspace of X. Let $\varepsilon \in [0,1)$. A vector $x \in X \setminus \{0\}$ is said to be *ε-perpendicular to* Y if

$$||x + y|| \geq (1 - \varepsilon)||x|| \quad \text{for all } y \in Y.$$

(i) Prove that if Y is a proper closed subspace of X, then for all $\varepsilon \in (0,1)$ there exists a *ε-perpendicular* vector.

(ii) Use (i) to prove that the closed unit ball of an infinite dimensional Banach space is not compact.

(iii) Let $L \in X'$ and set $Y := \mathrm{Ker}(L)$. Prove that Y admits a 0-perpendicular vector if and only if there exists $x \in \overline{B(0,1)}$ such that

$$||L|| = |L(x)|.$$

3. Let X be a Banach space and let $T \in \mathcal{L}(X; X')$.
(i) Assume that there exists $C > 0$ such that, for all $x \in X$,

$$\langle Tx, x\rangle_{X' \times X} \geq -C||Tx||^2_{X'}. \tag{1}$$

Prove that Ker$T \subset$ KerT^* (in the sense that if $x \in \mathrm{Ker}\, T$ then $J_X(x) \in \mathrm{Ker}\, T^*$).

(ii) Assume that Ker$T \subset$ KerT^* and that range$T$ is closed. Prove that there exists $C > 0$ such that (1) holds.

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4. (i) Let X be a normed space. State the Banach-Steinhaus Theorem.

(ii) Let $\{L_n\} \subset X'$ and $\{\varepsilon_n\} \subset (0, +\infty)$ be such that $\varepsilon_n \to 0$ and there exists $r > 0$ such that for every $x \in E$ with $||x|| < r$ there is $C(x) \in \mathbb{R}$ satisfying

$$L_n(x) \leq \varepsilon_n ||L_n|| + C(x) \quad \text{for all}\, n \in \mathbb{N}.$$

Prove that

$$\sup_n ||L_n|| < +\infty.$$

5. Let X be a Hilbert space, let E be a subspace of X, and consider a *monotone* map $A : E \to X'$, i.e.,

$$\langle A(x) - A(y), x - y \rangle \geq 0 \quad \text{for all}\, x, y \in E.$$

(i) Assume further that $A(0) = 0$, A is *maximal monotone*, i.e., $(\mathbb{I}+\tilde{A})(E) = X$, where $\tilde{A} : E \to X$ is defined by

$$\tilde{A}(x) := y(x) \text{ where}\, y(x) \text{ is the unique element of } X \text{such that}$$

$$\langle A(x), z \rangle = (z, y(x)) \text{ for all}\, z \in X.$$

Why is that there is such an element $y(x)$? Prove that E is dense in X.

(ii) Assume that $E = X$ and let $x_0 \in X$. Prove that there exist $R, C > 0$ such that if $x \in X$ is such that $||x - x_0|| < R$ then $||A(x)|| \leq C$.

(iii) Assume that $E = X$, X is separable, and for every $x, y \in X$

$$t \in \mathbb{R} \mapsto \langle A(x + ty), y \rangle \quad \text{is continuous at}\ t = 0.$$

Prove that $A : (X, ||\cdot||) \to (X', \sigma(X', X))$ is continuous.