Department of Mathematical Sciences
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BASIC EXAMINATION: FUNCTIONAL ANALYSIS

January 20, 2015, 4:30pm-6:30pm

Solve all five problems

1. (a) State the Open Mapping Theorem.

(b) Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces, and let $T \in \mathcal{L}(X;Y)$. Prove that if $T(X)$ has a finite complement then $T(X)$ is closed.

2. (a) State the Hahn-Banach Theorem (Second Geometric Form) over $\mathbb{R}$.

(b) Let X be a reflexive Banach space over $\mathbb{R}$ and let W be a closed linear subspace of X. Let C be a convex subset of X'. Define $\varphi : X \to \mathbb{R}$ by

$$\varphi(x) := \sup_{L\in C} L(x), \qquad x \in X.$$

(i) Prove that if $L \in \overline{W^\perp + C}$ then

$$\varphi(x) \geq L(x) \quad \text{for all } x \in W.$$

(ii) Conversely, show that if $L \in X'$ is such that

$$\varphi(x) \geq L(x) \qquad \text{for all } x \in W$$

then $L \in \overline{W^\perp + C}$.

3. (a) Let (X, τ) be a topological vector space. Give the precise definition of the weak topology $\sigma(X, X')$, and describe explicitely a balanced, convex local base of neighborhoods of the origin.

(b) Let $(X, ||\cdot||)$ be an infinite-dimensional Banach space such that X' is separable. Prove that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $||x_n|| = 1$ for all $n \in \mathbb{N}$, and $x_n \rightharpoonup 0$ with respect to the weak tolpology $\sigma(X, X')$.

4. (a) State the Banach-Steinhaus Theorem.

(b) Consider the space $C([0,1])$ with the L^p norm, $1 < p < +\infty$. Let $\{a_n\}$, $\{b_n\}$, be sequences in $(0,1)$ with $a_n < b_n$ for all $n \in \mathbb{N}$, and let $\{c_n\} \subset \mathbb{R}$. Define $L_n : C([0,1]) \to \mathbb{R}$ by

$$L_n(f) := c_n \int_{a_n}^{b_n} f(x)\, dx, \qquad \text{for } f \in C([0,1]).$$

(i) Prove that for all $n \in \mathbb{N}$

$$||L_n||_{C([0,1])'} = |c_n|(b_n - a_n)^{\frac{1}{p'}}.$$

(ii) Prove that $\sup_{n\in\mathbb{N}} |L_n(f)| < +\infty$ for all $f \in C([0,1])$ if and only if $\{c_n(b_n - a_n)\}$ is a bounded sequence.

(iii) Find $\{a_n\}, \{b_n\}, \{c_n\}$ such that $\sup_{n\in\mathbb{N}} |L_n(f)| < +\infty$ for every $f \in C([0,1])$ but $\sup_{n\in\mathbb{N}} ||L_n||_{C([0,1])'} = +\infty$.

(iv) Explain why (iii) is **not** is contradiction with the Banach-Steinhaus Theorem.

5. Let X be an infinite-dimensional Banach space.

(i) Let $x_0 \in X$ and $r \in \mathbb{R}$. Prove that the set

$$V := \{L \in X' : L(x_0) < r\}$$

is open with respect to $\sigma(X', X)$.

(ii) Prove that $(X', \sigma(X', X))$ is Hausdorff.

(iii) Prove that if $E \subset X'$ is bounded with respect to the norm $|| \cdot ||_{X'}$, then so is its closure with respect to the $\sigma(X', X)$ topology.

(iv) Prove that $(X', \sigma(X', X))$ can be written as a countable union of nowhere dense $\sigma(X', X)$ closed sets.

(v) Prove that $(X', \sigma(X', X))$ is complete.

(vi) Prove that $(X', \sigma(X', X))$ is not metrizable.