Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION: FUNCTIONAL ANALYSIS
September 2, 2014, Wean Hall 7201, 4:30pm-6:30pm

1. (i) State the Closed Graph Theorem.

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces.

(ii) Let $T : X \to Y$ be a linear and continuous operator such that Range(T) is closed. Prove that there exists $C > 0$ such that for every $y \in T(X)$ there exists $x \in X$ such that

$$y = T(x) \quad \text{and} \quad ||x||_X \le C||y||_Y.$$

(iii) Let $T : X \to Y$ be a linear operator such that for every sequence $\{x_n\} \subset X$

$$||x_n||_X \to 0 \Rightarrow T(x_n) \rightharpoonup 0 \quad \text{in } \sigma(Y, Y').$$

Prove that T is continuous, i.e., $T \in \mathcal{L}(X; Y)$.

2. Prove that if $(X, ||\cdot||)$ is a normed space over $\mathbb{R}$ such that X' is separable, then X is also separable.

3. (i) State and prove the Banach-Steinhaus Theorem for normed spaces.

(ii) Let $(X, ||\cdot||_X)$ be a Banach space over $\mathbb{R}$ and let $L_n, L \in X', n \in \mathbb{N}$, be such that $L_n \overset{\star}{\rightharpoonup} L$. Prove that

$$||L||_{X'} \le \liminf_{n\to\infty} ||L_n||_{X'} < +\infty.$$

4. (i) Give the definition of a compact operator between two normed spaces.

(ii) Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed spaces, and let $T : X \to Y$ be a linear compact operator. Prove that $T^\star$ is also compact.

5. Let $(X, (\cdot,\cdot))$ be a Hilbert space and let $T : X \to X$ be a linear, continuous operator such that

$$(Tx, x) \ge 0 \quad \text{for all } x \in X.$$

(i) Prove that $\text{Ker}(T) = (\text{Range}(T))^\perp$.

(ii) Prove that $\mathbb{I} + tT$ is bijective for all $t > 0$.