Discrete Mathematics: Basic Exam

August 25, 2025

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Name: ______________________________

Problem	Points
1	
2	
3	
4	
5	
Total	

Each problem is worth 20 points.

Problem 1 **20 points**

Let p be a prime, and let $\mathbb{F}_p$ denote the field with p elements.

(a) State the Chevalley–Warning theorem (about common zeros of polynomials over $\mathbb{F}_p$).

(b) Let $v_1, \ldots, v_{3p-1} \in \mathbb{F}_p^2$ with $\sum_{i=1}^{3p-1} v_i = 0$. Show that there is a set $I \subset [3p-1]$ of size $p-1$ or size p with $\sum_{i \in I} v_i = 0$.

Problem 2 **20 points**

Let $X \subset \mathbb{R}^2$ be a set of n points. Show that there is a red/blue coloring of the points in X such that in every axis-parallel rectangle R the number of red points deviates from the number of blue points by at most $c \cdot \sqrt{n \log(n)}$ for some universal constant $c > 0$.

Problem 3 **20 points**

Let F be a fixed 3-uniform hypergraph on six vertices with four hyperedges. Show that there is a 3-uniform hypergraph H on $[n]$ such that H has at least cn^2 hyperedges for some universal constant $c > 0$, any two distinct hyperedges of H intersect in at most one vertex, and the restriction of H to any six vertices does not contain a copy of F.

Problem 4 **20 points**

Let p be a prime, and let k be an integer with $2 < k < \frac{p}{2}$. Recall that a set $A \subset \mathbb{Z}/p\mathbb{Z}$ of integers modulo p is an arithmetic progression of length k and step size $d > 0$ if $A = \{a, a+d, a+2d, \ldots, a+(k-1)d\}$. Let P_k denote the number of arithmetic progression of length k and arbitrary step size that a fixed $x_0 \in \mathbb{Z}/p\mathbb{Z}$ is contained in.

(a) Let $\mathcal{F}$ be a set of arithmetic progressions of length k and fixed step size d in $\mathbb{Z}/p\mathbb{Z}$ that pairwise intersect. Show that for the size of $\mathcal{F}$ is at most k.

(b) Let $\mathcal{H}$ be a set of arithmetic progressions of length k and arbitrary step size in $\mathbb{Z}/p\mathbb{Z}$ that pairwise intersect. Show that the size of $\mathcal{H}$ is at most P_k.

Problem 5 **20 points**

Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for all $i \in [n]$. Show that of the 2^n sums $\sum_{i=1}^n \varepsilon_i x_i$ with $\varepsilon_i \in \{-1, +1\}$, at most $\binom{n}{\lceil \frac{n}{2} \rceil}$ lie in the interval $(-1, 1)$.