Discrete Mathematics: Basic Exam

January 23, 2025

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Name: _____

Problem	Points
1	
2	
3	
4	
5	
Total	

Each problem is worth 20 points.

Problem 1 **20 points**

A set $S \subset [n]$ is a *Sidon set* if for $x_1, x_2, x_3, x_4 \in S$ the equality $x_1 + x_2 = x_3 + x_4$ implies that $\{x_1, x_2\} = \{x_3, x_4\}$. Show that there is a Sidon set $S \subset [n]$ of size $|S| \geq cn^{1/3}$ for some constant $c > 0$.

Problem 2 **20 points**

Let G be a d-regular graph, that is, every vertex has degree d. Show that if $e \cdot \frac{k}{k^d} \cdot (d^2 + 1) < 1$ then there is a coloring of the vertices of G with k colors such that each vertex is adjacent to vertices of at least two distinct colors.

Problem 3 **20 points**

Prove that if $n \geq R(2k, 2k)$ and if we 2-color the edges of the complete bipartite graph $K_{n,n}$ then there is a monochromatic copy of $K_{k,k}$.

Recall that the Ramsey number $R(2k, 2k)$ is the smallest N such that for any 2-coloring of the edges of K_N there is a monochromatic clique of size $2k$.

Problem 4 **20 points**

(a) Let I_n denote the $n \times n$ identity matrix, and let J_n denote the $n \times n$ all ones matrix. Let S be an $(n \times n)$-matrix with $S + S^T = J_n - I_n$. Show that if for some $x \in \mathbb{R}^n$ we have that $Sx = 0$ and $\sum_i x_i = 0$, then $x = 0$ and deduce that S has rank at least $n - 1$.

(b) Show that if the edge set of K_n is the disjoint union of the edge set of m complete bipartite graphs, then $m \geq n - 1$ by writing $J_n - I_n$ as a sum of m matrices of rank one and their transpose matrices.

Problem 5 **20 points**

Let $A_1, \ldots, A_m \subset [n]$. Assume that there are two integers k_1 and k_2 such that for $i \neq j$ the symmetric difference $(A_i \setminus A_j) \cup (A_j \setminus A_i)$ has size k_1 or k_2. Show that $m \leq cn^2$ for some constant c.