Discrete Mathematics: Basic Exam
January 25, 2023

Do not flip the page until instructed.

Name: ________________________

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Each problem is worth 10 points.
Problem 1 10 points

For $B \subseteq \mathbb{Z}$ let $B+B = \{b_1 + b_2 : b_1, b_2 \in B, b_1 \neq b_2\}$. Fix an integer $k \geq 2$. Show that there is an integer $N$ such that for any $C \subseteq \mathbb{Z}$ and any subset $A \subseteq \mathbb{Z}$ with $|A| \geq N$ there is a subset $B \subseteq A$ of cardinality $k$ such that either $B+B \subseteq C$ or $B+B$ is disjoint from $C$.

Problem 2 10 points

Prove that for each $n \in \mathbb{N}$ there is a bipartite graph on $2n$ vertices, without 4-cycles as subgraphs, and with at least $\Omega(n^{4/3})$ edges.

Problem 3 10 points

(a) State the Cauchy–Davenport theorem.
(b) Let $p$ be a prime and let $n \in \mathbb{N}$. Let $A_1, \ldots, A_n \subseteq \mathbb{Z}/p$ be nonempty subsets with

$$|A_1| + |A_2| + \cdots + |A_n| = p + n - 1.$$ 

Show that $A_1 + \cdots + A_n = \mathbb{Z}/p$.

Problem 4 10 points

Fix an integer $k \geq 2$. Show that for any integer $n > k^{k+1}$ and any $k$-coloring of $[n] \times [n]$ there is a monochromatic rectangle. (A rectangle is a set of the form $\{(x,y), (x+a,y), (x,y+b), (x+a,y+b)\}$ for $a, b > 0$.)

Problem 5 10 points

Let $G$ be a graph with maximum degree $\Delta$. Let $k \geq 2e\Delta$, and let the vertex set of $G$ be partitioned into $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_q$ with $|V_i| \geq k$ for all $i \in [q]$. Show that there is an independent set with one vertex in each $V_i$.

Recall that a subset $A$ of vertices is an independent set if no two vertices in $A$ are connected by an edge.

Problem 6 10 points

Let $p \geq 3$ be a prime, $m = \frac{p-1}{2}$. Let $d_1, \ldots, d_m \in \mathbb{F}_p \setminus \{0\}$. Show that there exist pairwise distinct $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathbb{F}_p \setminus \{0\}$ with $x_i - y_i = d_i$ for all $i \in [m]$.

Here $\mathbb{F}_p$ denotes the field with $p$ elements. Use without proof that the coefficient of $\prod_{i=1}^{m} x_i^{2^{m-2}}$ in $g(x_1, \ldots, x_m) = \prod_{i<j} (x_i - x_j)^4$ is $\frac{(2m)!}{2^m}$. Apply the combinatorial Nullstellensatz to a suitable polynomial (obtained from $g$ by modifying its linear factors) that has the same coefficient of $\prod_{i=1}^{m} x_i^{2^{m-2}}$ as $g$. 