Discrete Mathematics: Basic Exam

January 25, 2023

Do not flip the page until instructed.

Name: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

Each problem is worth 10 points.

Problem 1 **10 points**

For $B \subseteq \mathbb{Z}$ let $B\hat{+}B = \{b_1 + b_2 \ : \ b_1, b_2 \in B, \ b_1 \neq b_2\}$. Fix an integer $k \geq 2$. Show that there is an integer N such that for any $C \subseteq \mathbb{Z}$ and any subset $A \subseteq \mathbb{Z}$ with $|A| \geq N$ there is a subset $B \subseteq A$ of cardinality k such that either $B\hat{+}B \subseteq C$ or $B\hat{+}B$ is disjoint from C.

Problem 2 **10 points**

Prove that for each $n \in \mathbb{N}$ there is a bipartite graph on $2n$ vertices, without 4-cycles as subgraphs, and with at least $\Omega(n^{4/3})$ edges.

Problem 3 **10 points**

(a) State the Cauchy–Davenport theorem.

(b) Let p be a prime and let $n \in \mathbb{N}$. Let $A_1, \ldots, A_n \subseteq \mathbb{Z}/p$ be nonempty subsets with

$$|A_1| + |A_2| + \cdots + |A_n| = p + n - 1.$$

Show that $A_1 + \cdots + A_n = \mathbb{Z}/p$.

Problem 4 **10 points**

Fix an integer $k \geq 2$. Show that for any integer $n > k^{k+1}$ and any k-coloring of $[n] \times [n]$ there is a monochromatic rectangle. (A rectangle is a set of the form $\{(x, y), (x+a, y), (x, y+b), (x+a, y+b)\}$ for $a, b > 0$.)

Problem 5 **10 points**

Let G be a graph with maximum degree Δ. Let $k \geq 2e\Delta$, and let the vertex set of G be partitioned into $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_q$ with $|V_i| \geq k$ for all $i \in [q]$. Show that there is an independent set with one vertex in each V_i.

Recall that a subset A of vertices is an *independent set* if no two vertices in A are connected by an edge.

Problem 6 **10 points**

Let $p \geq 3$ be a prime, $m = \frac{p-1}{2}$. Let $d_1, \ldots, d_m \in \mathbb{F}_p \setminus \{0\}$. Show that there exist pairwise distinct $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathbb{F}_p \setminus \{0\}$ with $x_i - y_i = d_i$ for all $i \in [m]$.

Here $\mathbb{F}_p$ denotes the field with p elements. Use without proof that the coefficient of $\prod_{i=1}^m x_i^{2m-2}$ in $g(x_1, \ldots, x_m) = \prod_{i<j}(x_i - x_j)^4$ is $\frac{(2m)!}{2^m}$. Apply the combinatorial Nullstellensatz to a suitable polynomial (obtained from g by modifying its linear factors) that has the same coefficient of $\prod_{i=1}^m x_i^{2m-2}$ as g.