21–701 Qualifying Exam

January 16, 2020

Name:

This exam consists of five problems and six pages, including this front page. You have three hours.

Good Luck!

Problem #	Max pts	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1 **20 points**

Let $p \in (0, 1]$ be constant. Show that with a probability approaching 1 the graph $G(n, p)$ has the property that every pair of its vertices has a common neighbor, i.e., a vertex, adjacent to both of them.

Problem 2 **20 points**

Show that the probability that $G(n, \frac{1}{2})$ has a bipartite subgraph with more than $\frac{1}{8}n^2 + n^{3/2}$ edges tends to zero.

Hint: Consider all possible bipartitions of the vertex set.

Problem 3 **20 points**

Let $c\colon \binom{\mathbb{N}}{2} \to \mathbb{N}$ be a coloring of pairs of natural numbers by natural numbers. Further assume that there is an $M \in \mathbb{N}$ such that at most M pairs receive the same color, that is,

$$|\{x \in \binom{\mathbb{N}}{2} \ : \ c(x) = i\}| \leq M$$

for every $i \in \mathbb{N}$. Show that there is an infinite $H \subset \mathbb{N}$ such that the elements of $\binom{H}{2}$ receive pairwise distinct colors by c.

Hint: Totally order $\binom{\mathbb{N}}{2}$ in an arbitrary way and apply Ramsey's theorem to a suitably defined coloring.

Problem 4 **20 points**

Let $c\colon \mathbb{N} \to [r]$ be an r-coloring of $\mathbb{N}$. Show that there is a 3-term arithmetic progression

$$\{a, a+d, a+2d\},$$

with $d \neq 0$, which is monochromatic with respect to c, and such that each of its three elements has an even number of not necessarily distinct prime divisors. (For example, $124 = 2^2 \cdot 31$ has an odd number of not necessarily distinct prime divisors.)

Problem 5 **20 points**

Given a set of lines in the plane, a point is called r-*rich* if it is contained in at least r of the lines. Show that for r sufficiently large the number of r-rich points among any L lines in the plane is $O(L^2/r^3 + L/r)$. (This is true for any $r \geq 2$, but the sufficiently large assumption might simplify the proof.)