Discrete Mathematics
Qualifying Exam 2019

Do not flip the page until instructed.

Name: ____________________________

Problem	Points
1	
2	
3	
4	
Total	

Each problem is worth 10 points.

No books, notes, or other external help is permitted.

*This is a **180** minute long test.*

1. A game is played between Alice and Bob. At start the elements of $[n] = \{1, 2, \ldots, n\}$ are unclaimed. Each turn Alice chooses a number d, and Bob claims a subset of $S \subset [n]$ that contains no two integers whose difference is d. The game is over when all elements of $[n]$ are claimed by Bob.

   Bob's goal is to finish the game quickly, whereas Alice's goal is to prolong the game.

   (a) (5 points) Show that Bob can finish the game in $O(\log n)$ steps.

   (b) (5 points) Show that Alice can make the game last at least $\Omega(\log \log n)$ steps.

2. (10 points) Suppose $\{a_n\}$ is a sequence that satisfies a linear recurrence, i.e., a relation of the form $a_n = \sum_{k=1}^{d} \gamma_k a_{n-k}$ for some constants $\gamma_1, \ldots, \gamma_d$. Let $b_n = a_{2n}$. Prove that b_n also satisfies a linear recurrence.

3. (10 points) Show that for each $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for every set S of m integers there is a k-coloring of $\mathbb{Z}$ such that every translate of S meets every color class.

4. (a) (5 points) State Szemerédi's regularity lemma and the triangle removal lemma.

   (b) (5 points) Assuming the regularity lemma, prove the triangle removal lemma.