Fall 2024

Differential Equations Basic Exam

You have 180 minutes to complete the five problems on the exam. If you make use of a major theorem, then cite the theorem explicitly.

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant L, and $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and satisfies $\sup_{z\in\mathbb{R}^n} |g(z) - f(z)| \le M$ for a constant $M > 0$. Let x and y be the maximal solutions of the respective initial value problems

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = x_0, \end{cases} \qquad \begin{cases} \dot{y}(t) = g(y(t)), \\ y(0) = y_0. \end{cases}$$

   (a) Show that $x(t)$ and $y(t)$ are defined for all $t \ge 0$. (You may take the local existence theory for granted.)

   (b) Show that for all $t \ge 0$,

$$|x(t) - y(t)| \le (|x_0 - y_0| + Mt)e^{Lt}.$$

2. (a) Let $U \subset \mathbb{R}^n$ be an open set and let $u \in C^2(U)$ solve $\Delta u = 0$ in U. Prove that there is a dimensional constant $C > 0$ such that

$$|\nabla u(x_0)| \le \frac{C}{r^{n+1}} \int_{B(x_0,r)} |u(x)|\, dx \tag{0.1}$$

   for any x_0 and $r > 0$ with $\overline{B(x_0, r)} \subset U$. Here $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

   Hint: First bound the left-hand side of (0.1) in terms of $\sup_{\partial B(x_0,r/2)} |u|$.

   (b) Use part (a) to deduce Liouville's theorem: if $u \in C^2(\mathbb{R}^n)$ is harmonic and bounded on $\mathbb{R}^n$, then u is constant.

3. Let $B_1 = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$. For $k \in \mathbb{N}$, let $u_k \in C^2_1(B_1 \times (0,\infty)) \cap C^0(\overline{B}_1 \times [0,\infty))$ solve

$$\begin{cases} (\partial_t - \Delta)u_k = 0 & \text{on } B_1 \times [0,\infty) \\ u_k = (1 - |x|^2)\sin(k|x|) & \text{on } B_1 \times \{t = 0\} \\ u_k = 0 & \text{on } \partial B_1 \times [0,\infty). \end{cases}$$

   (a) Find a function $F \in C^0(\overline{B}_1 \times [0,\infty))$ satisfying the properties
   - $F = 0$ on $\partial B_1 \times (0,\infty)$,
   - $\sup_{x\in B_1} |F(x,t)| \le C \exp\{-\alpha t\}$ for some $C > 0$ and $\alpha > 0$,
   - $F(x,t) \ge u_k(x,t)$ in $B_1 \times (0,\infty)$ for all $k \in \mathbb{N}$.

   Justify why the third property holds.

   (b) Show that there exist dimensional constants $C > 0$ and $\alpha > 0$ such that the functions u_k satisfy the uniform estimate

$$|u_k(x,t)| \le C\mathrm{dist}(x, \partial B_1)e^{-\alpha t}$$

   for all $(x,t) \in \overline{B}_1 \times [0,\infty)$. Here $\mathrm{dist}(x,\Sigma) = \inf\{|x - y| : y \in \Sigma\}$.

4. Suppose that $u \in C^2(\mathbb{R}^n \times [0, \infty))$ solves

$$\partial_{tt}u - \Delta u + k(x)\partial_t u = 0 \qquad \text{on } \mathbb{R}^n \times (0, \infty).$$

for a smooth bounded function $k : \mathbb{R}^n \to \mathbb{R}$. Fix $x_0 \in \mathbb{R}^n$ and $T > 0$ and suppose $u_t(x, 0) = u(x, 0) = 0$ for all x in $B(x_0, T)$. Show that $u(x_0, T) = 0$.

5. Consider the initial value problem

$$\begin{cases} xu_x + tu_t + u^2 = 0, & \text{for } x \in \mathbb{R}, t > 1 \\ u(x, 1) = \sin^2(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Use the method of characteristics to find a classical (C^1) solution for this equation on $\mathbb{R} \times (1, \infty)$.