Winter 2024

Differential Equations Basic Exam

You have 180 minutes to complete the five problems on the exam. If you make use of a major theorem, then cite the theorem explicitly.

1. (a) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and suppose there is a constant $K > 0$ such that $f(x) \cdot x \leq K|x|^2$ for any $x \in \mathbb{R}^n$ (here $|\cdot|$ is the Euclidean norm and $a \cdot b$ is the Euclidean inner product). Prove that for any $x_0 \in \mathbb{R}^n$, the solution to

$$\begin{cases} \dot{x}(t) = f(x) \\ x(0) = x_0 \end{cases}$$

exists for all $t \geq 0$ and satisfies $|x(t)| \leq |x_0| e^{Kt}$.

(b) Show that any solution $X(t) = (x(t), y(t))$ to

$$\begin{cases} \dot{x} &= 10x + 3y - xy \\ \dot{y} &= x - 8y + x^2 \end{cases}$$

exists for all $t \geq 0$ and satisfies $|X(t)| \leq |X(0)| e^{12t}$ for all $t \geq 0$.

2. Let $n = 2$ and suppose $u \in C^2(\mathbb{R}^2)$ satisfies $-\Delta u \leq 0$ on $\mathbb{R}^2$ and is bounded above on $\mathbb{R}^2$.

(a) Prove that

$$\sup_{x \in \mathbb{R}^2 \setminus B(0,1)} u(x) = \sup_{x \in \partial B(0,1)} u(x).$$

Here $B(0,1) = \{x \in \mathbb{R}^2 : |x| < 1\}$.

Hint: First prove the property for $v^\varepsilon = u + \varepsilon \Phi$ where Φ is the fundamental solution.

(b) Deduce that u is constant.

(c) A statement analogous to part (b) is false in higher dimensions. Indeed, describe (either by an explicit representation formula or by describing a construction) an example of a non-constant function $u \in C^2(\mathbb{R}^n)$ for some $n \geq 3$ that is subharmonic and bounded above.

3. Let $U \subset \mathbb{R}^n$ be an open bounded set with smooth boundary.

(a) Let $u \in C^2_1(U \times (0, \infty)) \cap C(\overline{U} \times [0, \infty))$ be a solution to

$$\begin{cases} (\partial_t - \Delta)u + u = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times (0, \infty) \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

for some $g \in C_0(U)$. Prove that for all $t \geq 0$,

$$\int_U u(x,t)^2 \, dx \leq e^{-2t} \int_U g^2(x,t) \, dx$$

(b) True or false: the same statement holds when the equation above is replaced by $(\partial_t - \Delta)u - u = 0$. Prove the statement or give a counterexample.

4. (a) Give the general expression for a solution to the wave equation

$$\begin{cases} (\partial_{tt} - \Delta)u(x,t) = f(x,t) & \text{for } x \in \mathbb{R}^3, t > 0 \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^3 \end{cases}$$

where f, g and h are smooth functions.

(b) Write an explicit expression (involving no integrals) for the solution to

$$\begin{cases} (\partial_{tt} - \Delta)u(x,t) = (e \cdot x)^2 & \text{for } x \in \mathbb{R}^3, t > 0 \\ u(x,0) = 0,\ u_t(x,0) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

where e is a unit vector in $\mathbb{R}^3$.

5. Consider the conservation law

$$u_t + (u^2/2)_x = 0, \qquad u(x,0) = u_0(x)$$

for $x \in \mathbb{R}$ and $t \geq 0$, where

$$u_0(x) = \begin{cases} 0 & \text{for } x < -1, \\ x+1 & \text{for } x \in (-1,1) \\ 0 & \text{for } x \geq 1 \end{cases}$$

Find a weak solution valid for all $t \geq 0$.