21-632: Basic Exam

Thursday, January 20, 2022, 6:30 - 9:30pm

1	2	3	4	5	total
10 pts	10 pts	10 pts	10 pts	10 pts	50 pts

Please read the following instructions carefully:

- You may not use any books, notes, or calculators.
- Switch off any electronic devices and put them in your bag (moblie phones, tablets, etc.)
- Exam duration: **180 minutes**
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- You may use theorems that we proved in class but you need to clearly state why the assumptions to apply these are satisfied and the results they imply.
- Follow any additional instructions as provided in Po-Shen Loh's email.

Good luck!

Notation

- Unless otherwise stated, $|\cdot|$ denotes the Euclidean vector norm in $\mathbb{R}^n$ ($n \in \mathbb{N}$): For a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$.

- For a function $u = u(t)$ depending on a variable $t \in \mathbb{R}$ (or a subset of $\mathbb{R}$), $u'(t)$ denotes the derivative with respect to that variable

$$u'(t) = \frac{du(t)}{dt}.$$

- For a function $u = u(t,x) = u(t, x_1, \dots, x_n)$ ($x = (x_1, \dots, x_n) \in \mathbb{R}^n$) depending on several variables, its partial derivatives with respect to the variables are denoted by

$$\frac{\partial u}{\partial t}(t,x) = \partial_t u(t,x) = u_t(t,x), \quad \frac{\partial u}{\partial x_j}(t,x) = \partial_{x_j} u(t,x) = u_{x_j}(t,x), \quad j = 1, \dots, n.$$

  We use powers to indicate we apply several partial derivatives, e.g., $\frac{\partial^2 u}{\partial t^2} = \partial_t^2 u = \partial_t \partial_t u$ or $\frac{\partial^3 u}{\partial x_j^3} = \partial_{x_j}^3 = \partial_{x_j} \partial_{x_j} \partial_{x_j}$.

- For a scalar valued function $u = u(x)$ (depending on space) or a function $u = u(t,x)$ (depending on space and time), we denote the gradient with respect to the spatial variables by $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ and the Laplace operator

$$\Delta u = \sum_{j=1}^n \partial_{x_j}^2 u.$$

- For a set $U \subset \mathbb{R}^n$, $n, m \in \mathbb{N}$ we denote

$$\begin{aligned}
C^0(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \mid u \text{ is continuous}\} \\
C^k(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \mid u \text{ is } k \text{ times continuously differentiable}\} \\
C^\infty(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \mid u \text{ is infintively often continuously differentiable}\} \\
C_c^k(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \mid u \in C^k(U,\mathbb{R}^m) \text{ and compactly supported in } U\} \\
C_c^\infty(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \mid u \in C^\infty(U,\mathbb{R}^m) \text{ and compactly supported in } U\}
\end{aligned}$$

  We say f is smooth if $f \in C^\infty(U;\mathbb{R}^m)$ and write $C^k(U) = C^k(U;\mathbb{R})$ etc.

- A function $f : U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$ a set, $n, m \in \mathbb{N}$) is Lipschitz continuous if there exists a positive real constant K such that for all $x_1, x_2 \in U$,

$$|f(x_1) - f(x_2)| \le K|x_1 - x_2|.$$

Problem 1: ODEs

Consider the autonomous ODE

$$\begin{aligned} u'(t) &= f(u(t)), \quad t \in I \subset \mathbb{R}, \\ u(t_0) &= u_0, \quad t_0 \in I, \end{aligned} \tag{1}$$

$u : I \mapsto \mathbb{R}^n$, $n \in \mathbb{N}$, where I is an interval.

a) Find an example of the function f (and initial condition u_0) for which the maximal interval of existence for the solution u of (1) is bounded.

b) Find an example of a function f (and initial condition u_0) for which the solution u of (1) is not unique.

c) Describe conditions on f (and initial conditions u_0) so that the solution u is unique and exists for all time.

For each part, justify your answer.

Problem 2: Elliptic equations

Given $0 < R_1 < R(t)$ let $U(t) = B_{R(t)}(0) \backslash \overline{B}_{R_1}(0) \subset \mathbb{R}^3$ and let $u(t, \cdot) \in C^2(\overline{U(t)})$ solve

$$\begin{aligned} \Delta u(t,x) &= 0, \quad R_1 < r < R(t), \quad r = |x|, \\ u(t,x) &= 1, \quad |x| = R_1, \\ u(t,x) &= 0, \quad |x| = R(t), \end{aligned} \tag{3}$$

In addition, assume that $R(t)$ satisfies

$$\frac{dR}{dt} = -\frac{\partial u}{\partial r}(r = R)$$

with initial condition $R(0) = R_0$ where $R_0 > R_1$.

a) Find the solution $u(t,x)$.

b) Find an ODE for the outer radius $R(t)$ (explicitly expressed in terms of $R(t)$).

Problem 3: Parabolic equations

Let $0 < L < \infty$.

a) Show that for functions $u \in C^1([0, L])$ with $\int_0^L u(x)dx = 0$,

$$\int_0^L |u(x)|^2 dx \leq C(L) \int_0^L |u'(x)|^2 dx,$$

where $C(L) > 0$ depends on L but not on u.

Hint: Use the fundamental theorem of calculus.

b) Let $0 < p(x) \in C^\infty([0,L])$ and $\varphi \in C^\infty([0,L])$. Consider the solution $u \in C^2([0,\infty) \times [0,L])$ of the equation

$$\begin{aligned} &u_t = \partial_x(p(x)\partial_x u), \quad x \in (0,L),\, t > 0,\\ &u(0,x) = \varphi(x), \quad x \in (0,L),\\ &\partial_x u(t,0) = \partial_x u(t,L) = 0, \quad t > 0. \end{aligned}$$

What is the limit of $u(t,x)$ as $t \to \infty$? Justify your answer.

Problem 4: Wave equation

Let $u = u(t,x) \in C^2([0,\infty) \times \mathbb{R}; \mathbb{R})$ solve

$$\begin{aligned} &u_{tt} + 2u_{tx} - u_{xx} + a(t,x)u_x = 0, \quad t > 0,\, x \in \mathbb{R}\\ &u(0,x) = f(x), \quad u_t(0,x) = g(x), \quad x \in \mathbb{R}, \end{aligned}$$

where f and g are smooth and compactly supported functions and a is a smooth and bounded function. Show that C^2-solutions of this problem are unique.

Problem 5: Characteristics

a) Solve $e^x u_x(x,y) + u_y(x,y) = u(x,y)$ with $u(x,0) = g(x)$.

b) Solve $x^2 u_x(x,y) + y^2 u_y(x,y) = (u(x,y))^2$ with $u(x,y) = 1$ for $y = 2x$.