Department of Mathematical Sciences Carnegie Mellon University

Basic Examination: Differential Equations September 2021

- This test is **closed book**: No books, notes, or access to any other relevant materials (including Internet consultation) are permitted.
- You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).
- You may use without proof *standard* results from the syllabus (not from homework) which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly **state** the result you are using.

Problem 1 : Assuming that $u : \mathbb{R}^3 \to \mathbb{R}$ is harmonic, prove that the vector field

$$v(x) = (u + 2x \cdot Du)Du - x|Du|^2, \qquad x \in \mathbb{R}^3,$$

has zero divergence, then use this to prove that for every $r > 0$,

$$\int_{B(0,r)} |Du|^2 \, dx \geq r \int_{\partial B(0,r)} |Du|^2 \, dS \, .$$

Finally, deduce from this that the function

$$r \mapsto \frac{1}{r} \int_{B(0,r)} |Du|^2 \, dx \, , \qquad r > 0,$$

is non-decreasing. (Notation: Du is the gradient of u, and $B(0,r) = \{x \in \mathbb{R}^n : |x| \leq r\}$.)

Problem 2 : Let U be a bounded open set in $\mathbb{R}^n$ with smooth boundary, and let $u(x,t)$ be a smooth solution of

$$\begin{aligned} u_t - \Delta u + c(x,t)u &= 0, \qquad && x \in U,\ t > 0, \\ u(x,t) &= 0, && x \in \partial U,\ t > 0, \\ u(x,t) &= g(x), && x \in U,\ t = 0. \end{aligned}$$

Assume $c(x,t) \geq \gamma > 0$ for all $x \in U$, $t \geq 0$. Prove that for some constant $C > 0$,

$$u(x,t) \leq Ce^{-\gamma t} \quad \text{for all } x \in U,\ t > 0.$$

Problem 3 : Suppose $0 < s < 1$ and $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are C^2. Find a solution formula for the solution to the following boundary value problem for the wave equation in $\mathbb{R}^2$:

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad && t > sx, \\ u(x,t) &= \phi(x), && t = sx, \\ u_t(x,t) &= \psi(x), && t = sx. \end{aligned}$$

analogous to d'Alembert's formula for the case $s = 0$. Explicitly determine the solution $u(x,t)$ in the case $\phi(x) = 0$, $\psi(x) = \sin x$.

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Problem 4 : Suppose $u\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ is a smooth solution of $u_t + uu_x = 0$ that is periodic in x with period L, i.e., $u(x+L,t) = u(x,t)$. Show that

$$\max_x u(x,0) - \min_x u(x,0) \leq \frac{L}{T}.$$

Problem 5 : Consider a scalar conservation law

$$u_t + f_\varepsilon(u)_x = 0, \qquad x\in\mathbb{R},\ t>0,$$

with convex flux function given by $f_\varepsilon(u) = \sqrt{u^2+\varepsilon}$ where $\varepsilon > 0$, and initial data

$$u(x,0) = \begin{cases} 0, & x<0,\\ 1, & x>0. \end{cases}$$

(i) Find a discontinuous weak solution of this problem, and determine whether the Lax entropy condition holds at points of discontinuity.

(ii) Show that a continuous solution $u^\varepsilon(x,t)$ exists in the form of a centered rarefaction wave for $t>0$. Find $u^0(x,t) = \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ and show that u^0 is a weak solution of

$$u_t + |u|_x = 0, \qquad x\in\mathbb{R},\ t>0.$$