BASIC EXAMINATION: DIFFERENTIAL EQUATIONS FEBRUARY 2021

- This test is **closed book**: No books, notes, or access to any other relevant materials (including Internet consultation) are permitted.
- You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).
- You may use without proof *standard* results from the syllabus (not from homework) which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly **state** the result you are using.

Problem 1 : Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic in the unit ball $B(0,1)$. For $a, c \in (0,1)$ let

$$G(a,c) = \int_{\partial B(0,1)} u(az)u(cz)\, dS(z).$$

Show that $G(a,c) = G(b,b)$ where $b^2 = ac$. (Hint: Show $G(a, b^2/a)$ is independent of a.)

Problem 2 : Suppose $U \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Show that the initial boundary value problem

$$\begin{aligned} u_t &= \Delta u + \int_U u^2\, dx \quad \text{in } U \text{ for } t > 0, \\ u &= 0 \qquad\qquad\qquad \text{on } \partial U \text{ for } t > 0, \\ u(x,0) &= g(x) \qquad\qquad \text{for } x \in U, \end{aligned}$$

may have *at most one* solution $u \in C^2(\overline{U} \times [0,T])$, for any $T > 0$.

Problem 3 : Consider this initial value problem for the inhomogeneous wave equation:

$$\begin{aligned} u_{tt} - u_{xx} = f(x,t), &\qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0 &\qquad -\infty < x < \infty, \end{aligned}$$

where f is smooth. Let $L > 0$ and $T > 0$, and suppose

$$\begin{aligned} f(x, t+T) = f(x,t) &\quad \text{for all } x \in \mathbb{R} \text{ and } t > 0, \text{ and} \\ f(x,t) = 0 &\quad \text{whenever } |x| > L, \text{ for all } t > 0. \end{aligned}$$

(a) Use Duhamel's principle and d'Alembert's formula to derive a solution formula for u.

(b) Show that $u_t + u_x = 0$ when $x = L$, for all $t > 0$.

(c) Show $u_t(x, t+T) = u_t(x,t)$ for all $x \geq 0$, if t is sufficiently large (depending on x).

Problem 4 : Find a solution of

$$(u_x - y)^2 + u_y^2 - 1 = 0,$$

which is smooth and positive in the first quadrant of the unit disk in $\mathbb{R}^2$ and which vanishes for $y = 0$. Show that the (projected) characteristics are circular arcs.

Problem 5 : Consider the scalar conservation law

$$u_t + (u^3 - u)_x = 0, \qquad -\infty < x < \infty, \quad t > 0.$$

(a) Find all constants $a \in \mathbb{R}$ such that the initial value problem has a simple shock solution

$$u(x,t) = \begin{cases} u_- & x < st, \\ u_+ & x > st, \end{cases}$$

satisfying the Lax entropy condition, with

$$u(x,0) = \begin{cases} 1 & x < 0, \\ a & x > 0. \end{cases}$$

(b) Find all constants $b \in \mathbb{R}$ such that the initial value problem has a continuous centered rarefaction wave weak solution, with

$$u(x,0) = \begin{cases} b & x < 0, \\ -1 & x > 0. \end{cases}$$

(c) For the initial data

$$u(x,0) = \begin{cases} 1 & x < 0, \\ -1 & x > 0, \end{cases}$$

describe two different weak solutions, one of which consists of a combination of a centered rarefaction wave and a simple shock.