Differential Equations: Basic Exam

Tuesday, September 8, 2020, 6:30-9:30pm

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Please read the following instructions carefully:

- Write your name on all sheets.
- You may not use any books, notes, or calculators.
- Switch off any electronic devices and put them in your bag (mobile phones, tablets, etc.)
- Exam duration: **180 minutes**
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Do not use pencils but rather pens.

Good luck!
Notation

• Unless otherwise stated, \( | \cdot | \) denotes the Euclidean vector norm in \( \mathbb{R}^n \) (\( n \in \mathbb{N} \)): For a vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), \( |x| := (x_1^2 + \cdots + x_n^2)^{1/2} \).

• \( \mathbb{R}_{\geq 0} := \{ x \in \mathbb{R} \mid x \geq 0 \} \)

• For a function \( y = y(t) \) depending on a variable \( t \in \mathbb{R} \) (or a subset of \( \mathbb{R} \)), \( y'(t) \) denotes the derivative with respect to that variable
  \[
  y'(t) = \frac{dy(t)}{dt}.
  \]

• For a function \( u = u(t,x) = u(t,x_1, \ldots, x_n) \) (\( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \)) depending on several variables, its partial derivatives with respect to the variables are denoted by
  \[
  \frac{\partial u}{\partial t}(t,x) = \partial_t u(t,x) = u_t(t,x), \quad \frac{\partial u}{\partial x_j}(t,x) = \partial_{x_j} u(t,x) = u_{x_j}(t,x), \quad j = 1, \ldots, n.
  \]

We use powers to indicate we apply several partial derivatives, e.g., \( \frac{\partial^2 u}{\partial t^2} = \partial_t^2 u = \partial_t \partial_t u \) or \( \frac{\partial^3 u}{\partial x_j^3} = \partial_{x_j}^3 = \partial_{x_j} \partial_{x_j} \partial_{x_j} \).

• For a scalar valued function \( u = u(x) \) (depending on space) or a function \( u = u(t,x) \) (depending on space and time), we denote the gradient with respect to the spatial variables by \( Du = (\partial_{x_1} u, \ldots, \partial_{x_n} u) \) and the Laplace operator
  \[
  \Delta u = \sum_{j=1}^n \partial_{x_j}^2 u.
  \]

• For a set \( U \subset \mathbb{R}^n \), \( n, m \in \mathbb{N} \) we denote
  \[
  C^0(U; \mathbb{R}^m) = \{ u : U \to \mathbb{R}^m \mid u \text{ is continuous} \}
  \]
  \[
  C^k(U; \mathbb{R}^m) = \{ u : U \to \mathbb{R}^m \mid u \text{ is } k \text{ times continuously differentiable} \}
  \]
  \[
  C^\infty(U; \mathbb{R}^m) = \{ u : U \to \mathbb{R}^m \mid u \text{ is infinitely often continuously differentiable} \}
  \]
  \[
  C^k_c(U; \mathbb{R}^m) = \{ u : U \to \mathbb{R}^m \mid u \in C^k(U, \mathbb{R}^m) \text{ and compactly supported in } U \}
  \]
  \[
  C^\infty_c(U; \mathbb{R}^m) = \{ u : U \to \mathbb{R}^m \mid u \in C^\infty(U, \mathbb{R}^m) \text{ and compactly supported in } U \}
  \]

We say \( f \) is smooth if \( f \in C^\infty(U; \mathbb{R}^m) \) and write \( C^k(U) = C^k(U; \mathbb{R}) \) etc.

• A function \( f : U \to \mathbb{R}^m \) (\( U \subset \mathbb{R}^n \) a set, \( n, m \in \mathbb{N} \)) is Lipschitz continuous if there exists a positive real constant \( K \) such that for all \( x_1, x_2 \in U \),
  \[
  |f(x_1) - f(x_2)| \leq K|x_1 - x_2|.
  \]
Problem 1: ODEs
Consider an ODE $y' = f(t, y(t))$ where $f$ is continuous for all $(t, y) \in \mathbb{R}^2$. Assume an initial value problem
$$y' = f(t, y(t)), \quad y(0) = y_0,$$
has two distinct solutions on $[0, T]$ for some $T > 0$. In particular, assume that the two solutions are bounded and take different values at $t = T$. Show that the ODE has infinitely many such solutions.

Problem 2: Elliptic equations
Let $U \subset \mathbb{R}^n$ be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 1$.

a) Let $B_r(x) = \{ y \in \mathbb{R}^n \mid |x - y| < r \}$ the ball with radius $r > 0$ in $\mathbb{R}^n$ and $\partial B_r(x)$ its boundary. Prove that for any $\phi \in C^2(U)$,
$$r^{n-1} \frac{\partial}{\partial r} \left( r^{1-n} \int_{\partial B_r(x)} \phi(y) dS(y) \right) = \int_{B_r(x)} \Delta \phi(y) dy, \quad \text{for all } B_r(x) \subset U.$$

b) Now let $f: U \to \mathbb{R}$ a continuous positive function and consider the PDE
$$\Delta (u^2) = f, \quad x \in U,$$
$$u = 0, \quad x \in \partial U.$$

Does a solution $u \in C^2(U) \cap C^0(\overline{U})$ of this PDE exist?

Problem 3: Parabolic equations
Suppose that $u \in C^2([0, \infty) \times [0, 1])$ is a solution of the initial boundary value problem
$$u_t = u_{xx} + cu^2, \quad t > 0, 0 < x < 1,$$
$$u(0, x) = u_0(x), \quad 0 \leq x \leq 1,$$
$$u(t, 0) = u(t, 1) = 0, \quad t > 0,$$
where $c$ is a positive constant and $u_0 \in C^2([0, 1])$ with $u_0(0) = u_0(1) = 0$.

a) Show that
$$\sup_{x \in [0,1]} |u(t, x)|^2 \leq \int_0^1 |u_x(t, x)|^2 dx.$$

b) Show that
$$\frac{1}{2} \frac{d}{dt} \int_0^1 |u(t, x)|^2 dx \leq - \int_0^1 |u_x(t, x)|^2 dx \left( 1 - c \left( \int_0^1 |u(t, x)|^2 dx \right)^{1/2} \right).$$

c) If the initial data $u_0$ satisfies $\int_0^1 |u_0(x)|^2 dx < 1/c^2$, show that $u$ satisfies $\int_0^1 |u(t, x)|^2 dx < 1/c^2$ for all times.
d) If the boundary condition is changed to \( \partial_x u(t, x) = 0 \) at \( x = 0 \) and \( x = 1 \) (and same for the initial data), find a counterexample, i.e., find an initial data \( u_0 \) for which the solution blows up in finite time.

**Problem 4:** Wave equation

Find a closed form (similar to D’Alembert’s formula) of the solution \( u(t, x) \) of

\[
\begin{align*}
  u_{tt} - c^2 u_{xx} &= 0, & \text{for } t, x > 0, \\
  u(0, x) &= g(x), & \text{for } x > 0, \\
  u_t(0, x) &= h(x), & \text{for } x > 0, \\
  u_x(t, 0) &= \alpha(t), & \text{for } t \geq 0,
\end{align*}
\]

where \( g, h, \alpha \in C^2 \) satisfy \( \alpha(0) = g'(0) \) and \( \alpha'(0) = h'(0) \).

**Problem 5:** Scalar conservation laws

Consider the conservation law

\[
u_t + u^3 u_x = 0, \quad u(0, x) = u_0(x) \tag{3}
\]

for \( (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \) and initial data \( u_0 \).

a) Define the characteristics for (3).

b) For the initial data

\[
u_0(x) = \begin{cases} 
-2, & x < 0, \\
1, & x \geq 0.
\end{cases}
\]

find two different weak solutions of the above PDE.

c) What is an entropy condition for the above PDE?

d) Find the entropy solution for the given initial data.