# Differential Equations: Basic Exam

Tuesday, September 8, 2020, 6:30-9:30pm

| 1 | 2 | 3 | 4 | 5 | total |
|---|---|---|---|---|---|
| 10 pts | 10 pts | 10 pts | 10 pts | 10 pts | 50 pts |
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**Please read the following instructions carefully:**

- Write your name on all sheets.
- You may not use any books, notes, or calculators.
- Switch off any electronic devices and put them in your bag (moblie phones, tablets, etc.)
- Exam duration: **180 minutes**
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Do not use pencils but rather pens.

Good luck!

# Notation

- Unless otherwise stated, $|\cdot|$ denotes the Euclidean vector norm in $\mathbb{R}^n$ ($n \in \mathbb{N}$): For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$.
- $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \,|\, x \geq 0\}$
- For a function $y = y(t)$ depending on a variable $t \in \mathbb{R}$ (or a subset of $\mathbb{R}$), $y'(t)$ denotes the derivative with respect to that variable

$$y'(t) = \frac{dy(t)}{dt}.$$

- For a function $u = u(t,x) = u(t, x_1, \ldots, x_n)$ ($x = (x_1, \ldots, x_n) \in \mathbb{R}^n$) depending on several variables, its partial derivatives with respect to the variables are denoted by

$$\frac{\partial u}{\partial t}(t,x) = \partial_t u(t,x) = u_t(t,x), \quad \frac{\partial u}{\partial x_j}(t,x) = \partial_{x_j} u(t,x) = u_{x_j}(t,x), \quad j = 1, \ldots, n.$$

  We use powers to indicate we apply several partial derivatives, e.g., $\frac{\partial^2 u}{\partial t^2} = \partial_t^2 u = \partial_t \partial_t u$ or $\frac{\partial^3 u}{\partial x_j^3} = \partial_{x_j}^3 = \partial_{x_j}\partial_{x_j}\partial_{x_j}$.

- For a scalar valued function $u = u(x)$ (depending on space) or a function $u = u(t,x)$ (depending on space and time), we denote the gradient with respect to the spatial variables by $Du = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ and the Laplace operator

$$\Delta u = \sum_{j=1}^{n} \partial_{x_j}^2 u.$$

- For a set $U \subset \mathbb{R}^n$, $n, m \in \mathbb{N}$ we denote

$$\begin{aligned}
C^0(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \text{ is continuous}\} \\
C^k(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \text{ is } k \text{ times continuously differentiable}\} \\
C^\infty(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \text{ is infintively often continuously differentiable}\} \\
C_c^k(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \in C^k(U,\mathbb{R}^m) \text{ and compactly supported in } U\} \\
C_c^\infty(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \in C^\infty(U,\mathbb{R}^m) \text{ and compactly supported in } U\}
\end{aligned}$$

  We say $f$ is smooth if $f \in C^\infty(U;\mathbb{R}^m)$ and write $C^k(U) = C^k(U;\mathbb{R})$ etc.

- A function $f : U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$ a set, $n, m \in \mathbb{N}$) is Lipschitz continuous if there exists a positive real constant $K$ such that for all $x_1, x_2 \in U$,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|.$$

## Problem 1: ODEs

Consider an ODE $y' = f(t, y(t))$ where $f$ is continuous for all $(t, y) \in \mathbb{R}^2$. Assume an initial value problem

$$y' = f(t, y(t)), \quad y(0) = y_0,$$

has two distinct solutions on $[0, T]$ for some $T > 0$. In particular, assume that the two solutions are bounded and take different values at $t = T$. Show that the ODE has infinitely many such solutions.

## Problem 2: Elliptic equations

Let $U \subset \mathbb{R}^n$ be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 1$.

**a)** Let $B_r(x) = \{y \in \mathbb{R}^n \,|\, |x - y| < r\}$ the ball with radius $r > 0$ in $\mathbb{R}^n$ and $\partial B_r(x)$ its boundary. Prove that for any $\phi \in C^2(U)$,

$$r^{n-1} \frac{\partial}{\partial r} \left( r^{1-n} \int_{\partial B_r(x)} \phi(y) dS(y) \right) = \int_{B_r(x)} \Delta \phi(y) dy, \quad \text{for all } B_r(x) \subset U.$$

**b)** Now let $f : U \to \mathbb{R}$ a continuous positive function and consider the PDE

$$\begin{aligned} &\Delta(u^2) = f, \quad x \in U, \\ &u = 0, \quad x \in \partial U. \end{aligned}$$

Does a solution $u \in C^2(U) \cap C^0(\overline{U})$ of this PDE exist?

## Problem 3: Parabolic equations

Suppose that $u \in C^2([0, \infty) \times [0, 1])$ is a solution of the initial boundary value problem

$$\begin{aligned} u_t &= u_{xx} + cu^2, \quad t > 0, \, 0 < x < 1, \\ u(0, x) &= u_0(x), \quad 0 \leq x \leq 1, \\ u(t, 0) &= u(t, 1) = 0, \quad t > 0, \end{aligned}$$

where $c$ is a positive constant and $u_0 \in C^2([0, 1])$ with $u_0(0) = u_0(1) = 0$.

**a)** Show that

$$\sup_{x \in [0,1]} |u(t, x)|^2 \leq \int_0^1 |u_x(t, x)|^2 dx.$$

**b)** Show that

$$\frac{1}{2} \frac{d}{dt} \int_0^1 |u(t, x)|^2 dx \leq - \int_0^1 |u_x(t, x)|^2 dx \left( 1 - c \left( \int_0^1 |u(t, x)|^2 dx \right)^{1/2} \right).$$

**c)** If the initial data $u_0$ satisfies $\int_0^1 |u_0(x)|^2 dx < 1/c^2$, show that $u$ satisfies $\int_0^1 |u(t, x)|^2 dx < 1/c^2$ for all times.

**d)** If the boundary condition is changed to $\partial_x u(t,x) = 0$ at $x = 0$ and $x = 1$ (and same for the initial data), find a counterexample, i.e., find an initial data $u_0$ for which the solution blows up in finite time.

## Problem 4: Wave equation

Find a closed form (similar to D'Alembert's formula) of the solution $u(t,x)$ of

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \quad \text{for } t, x > 0, \\ &u(0,x) = g(x), \quad \text{for } x > 0, \\ &u_t(0,x) = h(x), \quad \text{for } x > 0, \\ &u_x(t,0) = \alpha(t), \quad \text{for } t \geq 0, \end{aligned}$$

where $g, h, \alpha \in C^2$ satisfy $\alpha(0) = g'(0)$ and $\alpha'(0) = h'(0)$.

## Problem 5: Scalar conservation laws

Consider the conservation law

$$u_t + u^3 u_x = 0, \quad u(0,x) = u_0(x). \tag{3}$$

for $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ and initial data $u_0$.

**a)** Define the characteristics for (3).

**b)** For the initial data

$$u_0(x) = \begin{cases} -2, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

find two different weak solutions of the above PDE.

**c)** What is an entropy condition for the above PDE?

**d)** Find the entropy solution for the given initial data.