Differential Equations: Basic Exam

Tuesday, January 21, 2020, 4:30-7:30pm

1	2	3	4	5	total
10 pts	10 pts	10 pts	10 pts	10 pts	50 pts

Notation

- Unless otherwise stated, $|\cdot|$ denotes the Euclidean vector norm in $\mathbb{R}^n$ ($n \in \mathbb{N}$): For a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$.
- $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \,|\, x \geq 0\}$
- For a function $y = y(t)$ depending on a variable $t \in \mathbb{R}$ (or a subset of $\mathbb{R}$), $y'(t)$ denotes the derivative with respect to that variable

$$y'(t) = \frac{dy(t)}{dt}.$$

- For a function $u = u(t,x) = u(t, x_1, \dots, x_n)$ ($x = (x_1, \dots, x_n) \in \mathbb{R}^n$) depending on several variables, its partial derivatives with respect to the variables are denoted by

$$\frac{\partial u}{\partial t}(t,x) = \partial_t u(t,x) = u_t(t,x), \quad \frac{\partial u}{\partial x_j}(t,x) = \partial_{x_j} u(t,x) = u_{x_j}(t,x), \quad j = 1, \dots, n.$$

We use powers to indicate we apply several partial derivatives, e.g., $\frac{\partial^2 u}{\partial t^2} = \partial_t^2 u = \partial_t \partial_t u$ or $\frac{\partial^3 u}{\partial x_j^3} = \partial_{x_j}^3 = \partial_{x_j} \partial_{x_j} \partial_{x_j}$.

- For a scalar valued function $u = u(x)$ (depending on space) or a function $u = u(t,x)$ (depending on space and time), we denote the gradient with respect to the spatial variables by $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ and the Laplace operator

$$\Delta u = \sum_{j=1}^{n} \partial_{x_j}^2 u.$$

- For a set $U \subset \mathbb{R}^n$, $n, m \in \mathbb{N}$ we denote

$$\begin{aligned}
C^0(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \text{ is continuous}\} \\
C^k(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \text{ is } k \text{ times continuously differentiable}\} \\
C^\infty(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \text{ is infintively often continuously differentiable}\} \\
C_c^k(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \in C^k(U,\mathbb{R}^m) \text{ and compactly supported in } U\} \\
C_c^\infty(U;\mathbb{R}^m) &= \{u : U \to \mathbb{R}^m \,|\, u \in C^\infty(U,\mathbb{R}^m) \text{ and compactly supported in } U\}
\end{aligned}$$

We say f is smooth if $f \in C^\infty(U;\mathbb{R}^m)$ and write $C^k(U) = C^k(U;\mathbb{R})$ etc.

- A function $f : U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$ a set, $n, m \in \mathbb{N}$) is Lipschitz continuous if there exists a positive real constant K such that for all $x_1, x_2 \in U$,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|.$$

Problem 1: ODEs

Let $f : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ be continuous and assume that its derivative $\partial_y f(t,y)$ with respect to the variable y is continuous and satisfies $\partial_y f(t,y) \leq k(t)$ for a continuous function $k : [0,\infty) \to \mathbb{R}$. We consider the ordinary differential equation

$$y'(t) = f(t, y(t)), \quad t \in [0, \alpha), \quad y(0) = y_0, \tag{1}$$

where $y_0 \in \mathbb{R}$ and $\alpha > 0$ such that the solution y of (1) exists in $[0, \alpha)$.

a) Show that if $y_1(t)$ and $y_2(t)$ for $t > 0$ are solutions of (1) with initial data y_0^1 and y_0^2 respectively, they satisfy

$$|y_1(t) - y_2(t)| \leq \exp\left(\int_0^t k(s)ds\right) |y_0^1 - y_0^2|.$$

b) Show that (1) has solutions defined for all positive times $t \in [0, \infty)$.

c) Now consider

$$y'(t) = f(t, y(t)) + F(t, y(t)), \quad t \in [0, \infty), \quad y(0) = y_0, \tag{2}$$

where $y_0 \in \mathbb{R}$, $f, F : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and the y-derivative of f is continuous and satisfies $\partial_y f(t,y) \leq m < 0$ and $f(t,0) = 0$ for all t, and $F(t,y)$ satisfies $|F(t,y)| \leq g(t)$ with $\int_0^\infty g(s)ds < \infty$. Show that under these conditions (2) has bounded solutions in $t \in [0, \infty)$.

Problem 2: Harmonic functions: A maximum principle

Suppose $u \in C^2(\mathbb{R}^2_+) \cap C^0(\overline{\mathbb{R}}^2_+)$ is harmonic in $\mathbb{R}^2_+$ and bounded in $\overline{\mathbb{R}}^2_+$. Prove the maximum principle

$$\sup_{\mathbb{R}^2_+} u = \sup_{\partial\mathbb{R}^2_+} u.$$

Here $\mathbb{R}^2_+$ is the half-space $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$ and $\overline{\mathbb{R}}^2_+$ its closure: $\overline{\mathbb{R}}^2_+ := \{(x,y) \in \mathbb{R}^2 \,|\, y \geq 0\}$.

Hint: Consider for $\epsilon > 0$ the function

$$v_\epsilon(x,y) = u(x,y) - \epsilon \log(x^2 + (y+1)^2).$$

Problem 3: Explicit formula for a parabolic equation

Find an explicit formula for the solution of

$$\begin{aligned} \partial_t u + \alpha \cdot Du - \beta \Delta u + \gamma u = f, &\quad t > 0,\, x \in \mathbb{R}^n \\ u = g, &\quad t = 0,\, x \in \mathbb{R}^n, \end{aligned}$$

where $\alpha \in \mathbb{R}^n$, $\mathbb{R} \ni \beta > 0$, $\gamma \in \mathbb{R}$, $g \in C_c^2(\mathbb{R}^n)$, and $f \in C_c^2([0,\infty) \times \mathbb{R}^n)$.

Problem 4: Wave equations

Let $f \in C_c^\infty(\mathbb{R}^n)$, $n \geq 1$ and let u be a smooth solution of

$$\begin{aligned} &\partial_t^2 u - \Delta u = 0, \quad t > 0, x \in \mathbb{R}^n, \\ &u(0,x) = 0, \quad u_t(0,x) = f(x), \quad t = 0,\, x \in \mathbb{R}^n. \end{aligned}$$

Let $v(t,x) = \int_0^t u(s,x)ds$. Prove that

$$\int_{\mathbb{R}^n} (\Delta v(t,x))^2 dx \leq 4 \int_{\mathbb{R}^n} (f(x))^2 dx.$$

Problem 5: A scalar conservation law with source term

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable convex function. Assume $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. Consider the balance law

$$\partial_t u + \partial_x f(u) = g(u), \quad t > 0,\, x \in \mathbb{R}, \quad u(0,x) = u_0(x), \quad t = 0,\, x \in \mathbb{R}. \tag{3}$$

a) Define the characteristics for (3).

b) What is the Rankine-Hugoniot (jump) condition along a discontinuity for weak solutions in this case?

c) Set $f(u) = u^2/2$, $g(u) = u$ and

$$u_0(x) = \begin{cases} 1, & x < 0, \\ -1, & x \geq 0. \end{cases}$$

Find a weak solution of (3) for this case.

d) What is an entropy condition for (3)?