Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination in Differential Equations

Instructions: Work all 5 problems. **Time allowed: 3 hours**

Problem 1 : Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant K, and $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and bounded, with $|g(z)| \le M$ for all $z \in \mathbb{R}^n$. Let x and y be the maximal solutions of the respective initial value problems

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = x_0, \end{cases} \qquad \begin{cases} \dot{y}(t) = f(y(t)) + g(y(t)), \\ y(0) = y_0. \end{cases}$$

Show that for all $t \ge 0$, $x(t)$ and $y(t)$ are defined and satisfy

$$|x(t) - y(t)| \le (|x_0 - y_0| + Mt)e^{Kt}.$$

Problem 2 : Assume $u\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is C^2, $u(x) = o(|x|^{-1})$ and $Du(x) = o(|x|^{-2})$ as $x \to 0$, and

$\Delta u = 0$ in $\mathbb{R}^3 \setminus \{0\}$.

(a) Suppose $\Omega \subset \mathbb{R}^3$ be any bounded smooth domain with $0 \in \Omega$, and v (but not u) is C^2 on $\bar{\Omega}$. Prove

$$\int_\Omega u\Delta v = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} v \right) dS\,.$$

(b) Given any $x \in \mathbb{R}^3 \setminus \{0\}$, take for granted that $\Delta v = 1$ in $R^3 \setminus \{x\}$ if

$$v(y) = \frac{r^2}{6} + \frac{c_1}{r} + c_2\,, \quad r = |y - x|, \quad c_1,\, c_2 \text{ constant.}$$

Using part (a), prove that whenever $R > |x| > \varepsilon > 0$ and $A(x, \varepsilon, R) = \{y \in \mathbb{R}^3 : \varepsilon < |y - x| < R\}$,

$$u(x) = \frac{1}{|A(x,\varepsilon,R)|} \int_{A(x,\varepsilon,R)} u\,dy\,. \qquad \text{(Here } |A| \text{ is the volume of } A.\text{)}$$

(c) Now prove that $c = \lim_{x\to 0} u(x)$ exists, and show that with the definition $u(0) = c$, u becomes harmonic on all of $\mathbb{R}^3$.

Problem 3 : Let $u_0 : \mathbb{R} \to \mathbb{R}$ be continuous, bounded and *odd*: $u_0(-x) = -u_0(x)$. Let $u(x, t)$ be the solution of the heat equation $u_t = u_{xx}$ for $x \in \mathbb{R}$, $t > 0$, satisfying $u(x, 0) = u_0(x)$, given by convolution of u_0 with the fundamental solution $\Phi(x, t)$.

Suppose $u_0(x) = \frac{1}{x^p}$ for all $x > 1$. For what values of $p > 1$ does

$$v(x) = \lim_{t\to\infty} t^{p/2} u(x\sqrt{t}, t)$$

exist in $\mathbb{R}$ for all x? Justify your answer. (Hints: Some p, but not all. Also, $(a+b)^2 = (a-b)^2 + 4ab$.)

Problem 4 : Solve for the entropy (weak) solution of the conservation law $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ for $x \in \mathbb{R}$, $t > 0$, with initial data

$$u(x,0) = \begin{cases} 1 - x & \text{for } 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Draw a graph of the characteristic curves and any shock curves in the (x,t) plane.

Problem 5 : Suppose $a : \mathbb{R} \to \mathbb{R}$ is smooth with $0 < a(x) < 1$ for all x. Let $u \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be C^2 and solve the PDE

$$u_{tt} = (a(x)u_x)_x$$

for all $x \in \mathbb{R}$, $t \in \mathbb{R}$, with initial data $u(x,0) = g(x), \quad u_t(x,0) = h(x)$. Suppose $g(x) = h(x) = 0$ whenever $x > 0$. Prove that $u(x,t) = 0$ whenever $x > t$, by using an energy method to bound

$$E_\lambda(t) := \frac{1}{2}\int_{\mathbb{R}} (u_t^2 + a(x)u_x^2)e^{\lambda x}\, dx$$

for arbitrary $\lambda > 0$. You may assume the function $x \to u(x,t)$ has compact support for each t.