Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination in Differential Equations

Time allowed: 3 hours

Problem 1 : Let $c, L, T > 0$ and assume $u : [0, L] \times [0, T] \to [0, \infty)$ is nonnegative and C^2 (i.e., derivatives on the domain's interior extend continuously to its closure) and satisfies

$$\partial_t u = \partial_x^2 u + c^2 u + u^2, \quad x \in (0, L), \quad t \in (0, T),$$
$$u(0, t) = u(L, t) = 0, \quad t \in [0, T].$$

Define an "energy" via $\quad E(t) = \int_0^L u(x, t)\phi(x)\, dx = \int_0^L (u\sqrt{\phi})\sqrt{\phi}\, dx,$

where $\phi(x) = a \sin(bx)$ with a, b chosen so $\phi(L) = 0$, $\phi(x) > 0$ on $(0, L)$, and $\int_0^L \phi(x)\, dx = 1$.
If $cL \geq \pi$, show $E^2 \leq dE/dt$. Deduce that necessarily $E(0)T < 1$.

Problem 2 : Consider the following scalar conservation law with flux $f(u) = -u^3$:

$$\partial_t u - \partial_x(u^3) = 0, \qquad (x, t) \in U = \mathbb{R} \times (0, \infty).$$

(a) Describe two *different* weak solutions on $\bar{U} = \mathbb{R} \times [0, \infty)$ that have the same initial data

$$u(x, 0) = u_0(x) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x > 0. \end{cases}$$

(b) Suppose $v_0 : \mathbb{R} \to \mathbb{R}$ is smooth and decreasing and satisfies

$$v_0(x) \to 1 \ \text{ as } x \to -\infty, \qquad v_0(x) \to 0 \ \text{ as } x \to +\infty.$$

Using the method of characteristics, show that the conservation law has a global solution $u \in C^1(U) \cap C(\bar{U})$ satisfying $u(x, 0) = v_0(x)$ for all $x \in \mathbb{R}$.

(c) Given the solution u from part (b), determine the limit $\; v(x, t) = \lim_{\varepsilon \downarrow 0} u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \;$ for $x \in \mathbb{R}$, $t > 0$.

Problem 3 : Suppose $a : \mathbb{R}^n \to \mathbb{R}$ is continuous, positive and bounded. Prove the following weak maximum principle: If $M > 0$ is constant, and $u : \mathbb{R}^n \to \mathbb{R}$ is bounded and C^2, and satisfies

$$u(x) - a(x)\Delta u(x) \leq M \quad \forall x \in \mathbb{R}^n, \quad \text{then} \quad u(x) \leq M \quad \forall x \in \mathbb{R}^n.$$

(Note: $\max_x u(x)$ need not exist, but if $\varepsilon > 0$ and $u_\varepsilon = u - \varepsilon|x|^2$, then $\max_x u_\varepsilon(x)$ always exists.)

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Problem 4 : Formulate Duhamel's principle for the solution to the inhomogeneous wave equation in one space dimension with vanishing initial data:

$$\partial_t^2 u - \partial_x^2 u = H(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \quad u(x,0) = 0, \quad \partial_t u(x,0) = 0.$$

Suppose next that $H(x,t) = e^{it} a(x)$ where $a(x) = 0$ for $|x| > 1$. Show that there is a function $b(x)$ and a constant c such that for every $x \in \mathbb{R}$,

$$\lim_{t \to \infty} u(x,t) - e^{it} b(x) = c.$$

Problem 5 : Show that in $\mathbb{R}^3$, the function

$$\Phi(x) = \frac{e^{-r}}{4\pi r}, \quad r = |x|,$$

is the fundamental solution to the PDE

$$u - \Delta u = f$$

That is, show that if f is smooth with compact support, then the convolution

$$u(x) = \int_{\mathbb{R}^3} \Phi(y) f(x-y)\, d^3 y$$

is a solution of the PDE. Note that only a brief justification is needed for the formula

$$\Delta u = \int_{\mathbb{R}^3} \Phi(y) \Delta_x f(x-y)\, d^3 y\,.$$