Basic Examination Syllabi 2019

General Topology

• Finite and infinite sets, Axiom of Choice, Zorn’s lemma
• Topological Spaces (basics of open/closed sets, bases, countability, product spaces, quotient spaces)
• Continuity
• Connectedness, Compactness, Tychonoff Theorem
• Separation Axioms, Urysohn Lemma and Tietze Extension Theorems, Urysohn Metrization Theorem
• Metric Spaces (Cauchy sequences, completeness, Baire Category Theorem, completion, separability, sequential compactness, total boundedness, compactness, Arzela-Ascoli Theorem, partition of unity)
• Brouwer’s Fixed Point Theorem

References

• S. Willard, General topology Addison-Wesley, 1970.

Functional Analysis

• Linear spaces: Hilbert spaces, Banach spaces, topological vector spaces
• Hilbert spaces: geometry, projections, Riesz Representation Theorem, bilinear and quadratic forms, orthonormal sets and Fourier series.
• Banach spaces: continuity of linear mappings, Hahn-Banach Theorem, uniform boundedness, open-mapping theorem. Closed operators, closed graph theorem.
• Dual spaces: weak and weak-star topologies (Banach-Alaoglu Theorem), reflexivity. Space of bounded continuous functions and its dual, dual of $L^p$, dual of $L^\infty$.
• Linear operators and adjoints: basic properties, null spaces and ranges. Compact operators. Sequences of bounded linear operators: weak, strong and uniform convergence.
• Introduction to spectral theory: Notions of spectrum and resolvent set of bounded operators, spectral theory of compact operators.

References

• H. Brezis, Analyse fonctionnelle, Theorie et applications, Masson, 1983.
• A. Friedman, Foundations of Modern Analysis, Dover, 1982.
Measure and Integration

- Outer measure, measure, $\sigma$-algebras, Carathéodory’s Extension Theorem
- Borel measures, Lebesgue measure
- Measurable functions, Lebesgue integral (Monotone Convergence Theorem, Fatou’s Lemma, Dominated Convergence Theorem)
- Modes of Convergence (Egoroff’s Theorem, Lusin’s Theorem)
- Product Measures (Fubini-Tonelli Theorems), $n$-dimensional Lebesgue integral
- Signed Measures (Hahn Decomposition, Jordan Decomposition, Radon-Nikodym Theorem, change of variables)
- Differentiation (Lebesgue Differentiation Theorem)
- $L^p$ Spaces (Hölder’s inequality, Minkowski’s inequality, completeness, equiintegrability (uniform integrability), Vitali’s convergence theorem)

References


Probability

- Probability spaces, random variables, expectation, independence, Borel-Cantelli lemmas;
- Kernels and product spaces, existence of probability measures on infinite product spaces, Kolmogorov’s zero-one law.
- Weak and strong laws of large numbers, ergodic theorems, stationary sequences.
- Conditional expectation: characterization, construction and properties. Relation to kernels, conditional distribution, density.
- Filtration, adapted and predictable processes, martingales, stopping times, upcrossing inequality and martingale convergence theorems, backward martingales, optional stopping, maximal inequalities
- Weak convergence of probability measures, characteristic functions of random variables, weak convergence in terms of characteristic functions. Central limit theorem, Poisson convergence, Poisson process.
- Large deviations, rate functions, Cramer’s Theorem.

References

Discrete Mathematics
The examination is based on syllabus for the graduate course Discrete Mathematics (21-701).

Probabilistic Combinatorics
• The probabilistic method; including first moment, alterations, second moment methods, and Rödl nibble.
• Lovász Local Lemma; including the Moser-Tardos algorithm.
• Correlation inequalities.
• Martingales and tight concentration; including the differential equations method for establishing dynamic concentration.
• Janson’s inequality.
• Branching processes and coupling.
• Erdő-Rényi random graph and the configuration model for generating random regular graphs
• Markov chains; including mixing time and path coupling.

References
• The Probabilistic Method, N. Alon and J. Spencer. Chapters 1-8, 10.
• Random Graphs, S. Janson, T. Luczak, and A. Rucinski. Chapters 1-5.
• Markov Chains and Mixing Times, D. Levin, Y. Peres and E. Wilmer. Chapters 1-5, 14.

Set Theory
The examination is based on syllabus for the graduate course Set Theory (21-602).

Algebra
The examination is based on syllabus for the graduate course Algebra (21-610)

Model Theory
The examination is based on syllabus for the graduate courses Model Theory (21-603).

Differential Equations
• Essentials of ODE: existence, uniqueness, continuous dependence on data, flows, stability and asymptotic stability, linearization
• Elliptic equations: fundamental solution to Laplace equation / Newton potential and solutions in $\mathbb{R}^n$, Green’s functions in balls and half-spaces, mean-value property for Laplace’s equations, properties of harmonic functions
• Parabolic equations: heat kernel and solutions of heat equation in $\mathbb{R}^n$, mean-value property for the heat equation, energy method
• Hyperbolic equations: wave equation solutions in $\mathbb{R}^n$ for $n = 1, 2, 3$, Duhamel’s principle, energy method, finite speed of propagation
• First order theory: transport equation, continuity equation, method of characteristics
• 1D conservation laws with convex flux, shocks and rarefactions, Rankine-Hugoniot condition

References
• Ordinary Differential Equations by J. Hale
• Ordinary differential Equations with Applications by C. Chiccone
• Partial Differential Equations by L.C. Evans (Chapters 1-3)
• Partial Differential Equations by F. John