Basic Examination Syllabi 2019

General Topology

- topological spaces, continuous maps, initial (weak) and final topologies
- connectedness, countability axioms, compactness
- separation axioms, Urysohn Lemma and Tietze Extension Theorem, Urysohn Metrization Theorem, paracompactness and partitions of unity
- nets, filters, ultrafilters, Tychonoff's theorem, Stone–Čech compactification
- homotopy (homotopy equivalence of spaces, contractible spaces, deformation retractions), fundamental group, group actions, covering spaces
- Brouwer's Fixed Point Theorem

References

- J. Munkres, *Topology* (2nd edition) Prentice Hall 2000.
- N. Bourbaki, *General Topology: Chapters 1–4* Elements of Mathematics, Springer-Verlag, 1998.
- J. Kelley, *General topology* D. Van Nostrand, 1955.
- S. Willard, *General topology* Addison-Wesley, 1970.

Functional Analysis

- Linear spaces: Hilbert spaces, Banach spaces, topological vector spaces
- Hilbert spaces: geometry, projections, Riesz Representation Theorem, bilinear and quadratic forms, orthonormal sets and Fourier series.
- Banach spaces: continuity of linear mappings, Hahn-Banach Theorem, uniform boundedness, open-mapping theorem. Closed operators, closed graph theorem.
- Dual spaces: weak and weak-star topologies (Banach-Alaoglu Theorem), reflexivity. Space of bounded continuous functions and its dual, dual of L^p, dual of L^∞.
- Linear operators and adjoints: basic properties, null spaces and ranges. Compact operators. Sequences of bounded linear operators: weak, strong and uniform convergence.
- Introduction to spectral theory: Notions of spectrum and resolvent set of bounded operators, spectral theory of compact operators.

References

- M. Reed and B. Simon, Methods of Mathematical Physics I: Functional Analysis (2nd edition), Academic Press, 1980.
- N. Dunford and J. T. Schwartz, Linear Operators. Part I: General Theory, Wiley Interscience, 1958.
- T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1980.
- H. Brezis, Analyse fonctionnelle, Theorie et applications, Masson, 1983.
- P. Lax, Functional Analysis, Wiley Interscience, 2002.
- W. Rudin, Functional Analysis, McGraw-Hill, 1973.
- A. Friedman, Foundations of Modern Analysis, Dover, 1982.

Measure and Integration

- Outer measure, measure, σ-algebras, Carathéodory's Extension Theorem
- Borel measures, Lebesgue measure
- Measurable functions, Lebesgue integral (Monotone Convergence Theorem, Fatou's Lemma, Dominated Convergence Theorem)
- Modes of Convergence (Egoroff's Theorem, Lusin's Theorem)
- Product Measures (Fubini-Tonelli Theorems), n-dimensional Lebesgue integral
- Signed Measures (Hahn Decomposition, Jordan Decomposition, Radon-Nikodym Theorem, change of variables)
- Differentiation (Lebesgue Differentiation Theorem)
- L^p Spaces (Hölder's inequality, Minkowskii's inequality, completeness, equiintegrability (uniform integrability), Vitali's convergence theorem)

References

- R.G. Bartle, *The Elements of Integration and Lebesgue Measure*, Containing a corrected reprint of the 1966 original, John Wiley & Sons, Inc., 1995.
- D.L. Cohn, *Measure Theory.* Reprint of the 1980 original, Birkhäuser Boston, Inc., 1993.
- E. DiBenedetto, *Real Analysis*, Birkhäuser, 2002.
- G.B. Folland, *Real analysis. Modern Techniques and Their Applications*, Second edition, John Wiley & Sons, 1999.
- I. Fonseca and G. Leoni, *Modern Methods in the Calculus of Variations: L^p Spaces*, Springer, 2007.
- M.M. Rao, *Measure Theory and Integration*, Second edition, Marcel Dekker, 2004.
- H.L. Royden, *Real analysis*, Third edition. Macmillan Publishing Company, 1988.
- W. Rudin, *Real and Complex Analysis*, Third edition. McGraw-Hill Book Co., 1987.

Probability

- Probability spaces, random variables, expectation, independence, Borel-Cantelli lemmas;
- Kernels and product spaces, existence of probability measures on infinite product spaces, Kolmogorov's zero-one law.
- Weak and strong laws of large numbers, ergodic theorems, stationary sequences.
- Conditional expectation: characterization, construction and properties. Relation to kernels, conditional distribution, density.
- Filtration, adapted and predictable processes, martingales, stopping times, upcrossing inequality and martingale convergence theorems, backward martingales, optional stopping, maximal inequalities
- Weak convergence of probability measures, characteristic functions of random variables, weak convergence in terms of characteristic functions. Central limit theorem, Poisson convergence, Poisson process.
- Large deviations, rate functions, Cramer's Theorem.

References

- Leo Breiman *Probability* (Corrected reprint of the 1968 original) SIAM, 1992.
- P. Billingsley *Probability and Measure* (3rd edition) John Wiley and Sons, 1995.
- R. Durrett *Probability: Theory and Examples* (3rd edition), Brooks/Cole 2005.
- D. Khoshnevisan *Probability*, AMS, 2007.
- D. Williams *Probability with Martingales* Cambridge University Press, 1991.

Discrete Mathematics

The examination is based on syllabus for the graduate course Discrete Mathematics (21-701).

Probabilistic Combinatorics

- The probabilistic method; including first moment, alterations, second moment methods, and Rödl nibble.
- Lovász Local Lemma; including the Moser-Tardos algorithm.
- Correlation inequalities.
- Martingales and tight concentration; including the differential equations method for establishing dynamic concentration.
- Janson's inequality.
- Branching processes and coupling.
- Erdős-Renyi random graph and the configuration model for generating random regular graphs
- Markov chains; including mixing time and path coupling.

References

- *The Probabilistic Method*, N. Alon and J. Spencer. Chapters 1-8, 10.
- *Random Graphs*, S. Janson, T. Luczak, and A. Rucinski. Chapters 1-5.
- *Markov Chains and Mixing Times*, D. Levin, Y. Peres and E. Wilmer. Chapters 1-5, 14.

Set Theory

The examination is based on syllabus for the graduate course Set Theory (21-602).

Algebra

The examination is based on syllabus for the graduate course Algebra (21-610)

Model Theory

The examination is based on syllabus for the graduate courses Model Theory (21-603).

Differential Equations

- Essentials of ODE: existence, uniqueness, continuous dependence on data, flows, stability and asymptotic stability, linearization
- Elliptic equations: fundamental solution to Laplace equation / Newton potential and solutions in $\mathbb{R}^n$, Green's functions in balls and half-spaces, mean-value property for Laplace's equations, properties of harmonic functions
- Parabolic equations: heat kernel and solutions of heat equation in $\mathbb{R}^n$, mean-value property for the heat equation, energy method
- Hyperbolic equations: wave equation solutions in $\mathbb{R}^n$ for $n = 1, 2, 3$, Duhamel's principle, energy method, finite speed of propagation
- First order theory: transport equation, continuity equation, method of characteristics
- 1D conservation laws with convex flux, shocks and rarefactions, Rankine-Hugoniot condition

References

- *Ordinary Differential Equations* by J. Hale
- *Ordinary differential Equations with Applications* by C. Chiccone
- *Partial Differential Equations* by L.C. Evans (Chapters 1-3)
- *Partial Differential Equations* by F. John