General Topology

- topological spaces, continuous maps, initial (weak) and final topologies
- connectedness, countability axioms, compactness
- separation axioms, Urysohn Lemma and Tietze Extension Theorem, Urysohn Metrization Theorem, paracompactness and partitions of unity
- nets, filters, ultrafilters, Tychonoff’s theorem, Stone–Čech compactification
- homotopy (homotopy equivalence of spaces, contractible spaces, deformation retractions), fundamental group, group actions, covering spaces
- Brouwer’s Fixed Point Theorem

References


Functional Analysis

- Linear spaces: Hilbert spaces, Banach spaces, topological vector spaces
- Hilbert spaces: geometry, projections, Riesz Representation Theorem, bilinear and quadratic forms, orthonormal sets and Fourier series.
- Banach spaces: continuity of linear mappings, Hahn-Banach Theorem, uniform boundedness, open-mapping theorem. Closed operators, closed graph theorem.
- Dual spaces: weak and weak-star topologies (Banach-Alaoglu Theorem), reflexivity. Space of bounded continuous functions and its dual, dual of $L^p$, dual of $L^\infty$.
- Linear operators and adjoints: basic properties, null spaces and ranges. Compact operators. Sequences of bounded linear operators: weak, strong and uniform convergence.
- Introduction to spectral theory: Notions of spectrum and resolvent set of bounded operators, spectral theory of compact operators.

References

Measure and Integration

- Outer measure, measure, $\sigma$-algebras, Carathéodory’s Extension Theorem
- Borel measures, Lebesgue measure
- Measurable functions, Lebesgue integral (Monotone Convergence Theorem, Fatou’s Lemma, Dominated Convergence Theorem)
- Modes of Convergence (Egoroff’s Theorem, Lusin’s Theorem)
- Product Measures (Fubini-Tonelli Theorems), n-dimensional Lebesgue integral
- Signed Measures (Hahn Decomposition, Jordan Decomposition, Radon-Nikodym Theorem, change of variables)
- Differentiation (Lebesgue Differentiation Theorem)
- $L^p$ Spaces (Hölder’s inequality, Minkowskii’s inequality, completeness, equiintegrability (uniform integrability), Vitali’s convergence theorem)

References


Probability

- Probability spaces, random variables, expectation, independence, Borel-Cantelli lemmas;
- Kernels and product spaces, existence of probability measures on infinite product spaces, Kolmogorov’s zero-one law.
- Weak and strong laws of large numbers, ergodic theorems, stationary sequences.
- Conditional expectation: characterization, construction and properties. Relation to kernels, conditional distribution, density.
- Filtration, adapted and predictable processes, martingales, stopping times, upcrossing inequality and martingale convergence theorems, backward martingales, optional stopping, maximal inequalities
- Weak convergence of probability measures, characteristic functions of random variables, weak convergence in terms of characteristic functions. Central limit theorem, Poisson convergence, Poisson process.
- Large deviations, rate functions, Cramer’s Theorem.

References

Discrete Mathematics
The examination is based on syllabus for the graduate course Discrete Mathematics (21-701).

Probabilistic Combinatorics

- The probabilistic method; including first moment, alterations, second moment methods, and Rödl nibble.
- Lovász Local Lemma; including the Moser-Tardos algorithm.
- Correlation inequalities.
- Martingales and tight concentration; including the differential equations method for establishing dynamic concentration.
- Janson’s inequality.
- Branching processes and coupling.
- Erdő-Renyi random graph and the configuration model for generating random regular graphs
- Markov chains; including mixing time and path coupling.

References

Set Theory
The examination is based on syllabus for the graduate course Set Theory (21-602).

Algebra
The examination is based on syllabus for the graduate course Algebra (21-610)

Model Theory
The examination is based on syllabus for the graduate courses Model Theory (21-603).

Differential Equations

- Essentials of ODE: existence, uniqueness, continuous dependence on data, flows, stability and asymptotic stability, linearization
- Elliptic equations: fundamental solution to Laplace equation / Newton potential and solutions in \( \mathbb{R}^n \), Green’s functions in balls and half-spaces, mean-value property for Laplace’s equations, properties of harmonic functions
- Parabolic equations: heat kernel and solutions of heat equation in \( \mathbb{R}^n \), mean-value property for the heat equation, energy method
- Hyperbolic equations: wave equation solutions in \( \mathbb{R}^n \) for \( n = 1, 2, 3 \), Duhamel’s principle, energy method, finite speed of propagation
- First order theory: transport equation, continuity equation, method of characteristics
- 1D conservation laws with convex flux, shocks and rarefactions, Rankine-Hugoniot condition

References
- *Ordinary Differential Equations* by J. Hale
- *Ordinary differential Equations with Applications* by C. Chiccone
- *Partial Differential Equations* by L.C. Evans (Chapters 1-3)
- *Partial Differential Equations* by F. John