

BASIC EXAM IN ALGEBRA

CMU SPRING 2026

Each problem is worth 20 points. Solve as many as you can. Good luck!

- (1). Let H be a proper subgroup of a group G . If G is finite, show that G is not a union of conjugates of H . Show by example that if G is infinite, then G can be a union of conjugates of H .
- (2). Let $\mathbb{Q}$ be the field of rational numbers.
- (a) Show that $\mathbb{Q}$, as an abelian group under addition, is not a direct sum of two nontrivial abelian groups.
 - (b) Show that $\mathbb{Q}_{>0}$, the multiplicative group of positive rational numbers, is a free abelian group. Does $\mathbb{Q}_{\neq 0}$, the multiplicative group of nonzero rational numbers, have $\mathbb{Q}_{>0}$ as a direct summand?
- (3). Recall that an *Artinian* ring R is a unital commutative ring satisfying the *descending chain condition*: every descending chain of ideals in R stabilizes.
- (1) Give an example, with proof, of an Artinian ring that is not a field.
 - (2) Show that an Artinian integral domain is a field. Conclude that every prime ideal of an Artinian ring is maximal.
- (4). For each item, provide an example with proof.
- An integer n and two $n \times n$ matrices A and B over a field such that: A and B are not similar but share the same minimal polynomial and characteristic polynomial.
 - A real 4×4 matrix A with no real eigenvalues such that A is not diagonalizable over $\mathbb{C}$. (Possibly helpful: $\mathbb{C} \simeq \mathbb{R}^2$ as $\mathbb{R}$ -vector spaces).
- (5). Let m be the smallest positive integer such that the m -th primitive root of unity ζ satisfies $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\zeta)$. Find m , and classify all intermediate field extensions $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\zeta)$. If you wish to use facts about cyclotomic extensions, you must prove them, but you may use the fundamental theorem of Galois theory without proof.