

BASIC EXAM IN ALGEBRA

CMU FALL 2025

Each problem is worth 20 points. Solve as many as you can. Good luck!

- (1). Show that a group of order p^2q is solvable, where p and q are distinct positive prime integers.
- (2). Recall that an *abelianization* of a group G is an abelian group G^{ab} along with a homomorphism $f : G \rightarrow G^{ab}$ satisfying the following universal property: For every abelian group A and a homomorphism $g : G \rightarrow A$, there exists a unique homomorphism $\tilde{g} : G^{ab} \rightarrow A$ such that $g = \tilde{g} \circ f$.
- Show that an abelianization of a group (as defined above) always exists.
 - Let G and H be groups. For each of the following statements, either prove it or give a counterexample. (Here, $\times$, $\ltimes$, $*$, and $\oplus$ denote the direct product, semidirect product, free product, and direct sum, respectively).
 - $(G \times H)^{ab}$ is isomorphic to $G^{ab} \times H^{ab}$.
 - Suppose G acts on H , and suppose H and $\text{Aut}(H)$ are abelian (so G^{ab} acts on H). Then, $(G \ltimes H)^{ab}$ is isomorphic to $G^{ab} \ltimes H^{ab}$.
 - $(G * H)^{ab}$ is isomorphic to $G^{ab} \oplus H^{ab}$.
- (3). Let $\varphi : \mathbb{Z}^4 \rightarrow \mathbb{Z}^4$ be a map of free $\mathbb{Z}$ -modules given by multiplication by the matrix

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

- Why are $\ker \varphi$ and $\text{coker } \varphi$ finitely generated?
 - Express $\ker \varphi$ and $\text{coker } \varphi$ as a direct sum of cyclic $\mathbb{Z}$ -modules.
- (4). Let $\mathbb{k}$ be a field, and $f(x) \in \mathbb{k}[x]$ a separable polynomial. Let F be the splitting field of f extending $\mathbb{k}$. Show that $\text{Aut}(F/\mathbb{k})$ acts faithfully on the set of roots of f . If f is irreducible, show further that the action is transitive. Show by example (with justification) that even if f is irreducible, the action need not be free.
- (5). Suppose α is a root of $x^3 + 2x + 1 \in \mathbb{Q}[x]$. Does $x^3 + x + 1$ have a root in $\mathbb{Q}(\alpha)$? (Possible hint: The discriminant of a depressed cubic $x^3 + ax + b$ is $-4a^3 - 27b^2$).