

Algebra basic exam, August 2022

180 minutes

Each of the five questions is worth the same.

1. Give an example of a module over a principal ideal domain that is not isomorphic to a direct sum of cyclic modules. Justify your example.
2. Let F/K be a quadratic field extension.
 - (a) Prove that there is $\alpha \in F$ such that $\alpha^2 \in K$ and $F = K(\alpha)$ if $\text{char } K \neq 2$.
 - (b) Does such an $\alpha \in F$ exist if $\text{char } K = 2$ and K is finite?
3. Prove that the free groups $F(x, y)$ and $F(x, y, z)$ are not isomorphic.
4.
 - (a) Define the terms ‘cyclotomic extension’ and ‘Galois extension’.
 - (b) Are all cyclotomic extensions Galois? Justify.
5. Let R be a Noetherian ring. Prove that $R[x]$ is also Noetherian.