Algebra basic exam, January 2022

180 minutes

Each of the five questions is worth the same.

1. Let K/F be a field extension with both fields K and F being algebraically closed, and $K \neq F$. Prove that there are infinitely many intermediate fields L between K and F.

2. Prove that there exists a non-abelian group of order $609 = 3 \cdot 7 \cdot 29$.

3. (a) Define the term *R-module*.

   (b) Is there an abelian group that cannot be given a structure of a $\mathbb{Q}$-module? Justify your answer.

4. Let $\mathbb{F}_q$ denote the finite field of cardinality q. Let $L = \mathbb{F}_9(x)$ and $K = \mathbb{F}_3(x)$. Is the field extension L/K Galois? If yes, what is the Galois group of the extension? If not, does this extension admit a Galois closure?

5. Let F be a field, and let $\prec$ be a monomial ordering on $F[x_1, \ldots, x_n]$.

   (a) Define the term *Gröbner basis with respect to $\prec$*.

   (b) Give an example of a Gröbner basis G (with respect to a monomial ordering of your choice) such that $G + 1 := \{g + 1 : g \in G\}$ is *not* a Gröbner basis (for any ideal).