Algebra basic exam, September 2021

180 minutes
Each of the five questions is worth the same.

1. (a) Define the term *unique factorization domain.*

   (b) Is the ring $\mathbb{Z}[x, y]$ unique factorization domain?

2. (a) A group G of order 35 acts on a set X of size 16. Show that the action has a fixed point.

   (b) Show that, up to isomorphism, there is a unique group of order 35.

   Hint/warning: The parts (a) and (b) are unrelated.

3. Let p be a prime, and let $\mathbb{F}_p$ a field with p element. Let $f \in \mathbb{F}_p[x]$ be a polynomial such that the map $c \mapsto f(c)$ is an automorphism of $\mathbb{F}_p$. Show that either $\deg f = 1$ or $\deg f \geq p$.

4. (a) Define the terms *splitting field* and *Galois group.*

   (b) What are all the possible Galois groups of the splitting field of $x^4 + n$ over $\mathbb{Q}$, for n an integer?

5. Let R be a commutative ring with 1.

   (a) Define the term *free R-module.*

   (b) Suppose M and N are free R-modules. Is $M \otimes_R N$ necessarily free?