ALGEBRA BASIC EXAM: SEPTEMBER 2020

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be rings with 1, and all ring homomorphisms are assumed to preserve 1.

(1) State and prove the Sylow theorem(s). Prove that if G is finite, $H \lhd G$ and P is a Sylow p-subgroup of H for some prime p then $G = HN_G(P)$, where $N_G(P)$ is the normaliser of P in G. Hint: Let $g \in G$,and consider P^g.

(2) Define the concept of *nilpotent group*. Prove that if G is a nilpotent group and $M < G$ then $M < N_G(M)$. Hint: Think about the lower (descending) central series of G.

(3) Define the terms *algebraic extension*, *separable extension*, *normal extension*, *splitting field extension*, *Galois extension*. State some version of the theorem on uniqueness of splitting field extensions, and use it to prove that splitting field extensions are normal.

(4) State the Fundamental Theorem of Galois theory. Let $\alpha = \sqrt[4]{2}$ and $\zeta = e^{\pi i/4}$, and let $F = \mathbb{Q}(\alpha, \zeta)$. Determine with proof:
   (a) $[\mathbb{Q}(\alpha) : \mathbb{Q}]$, $[\mathbb{Q}(\zeta) : \mathbb{Q}]$ and $[F : \mathbb{Q}]$.
   (b) The structure of $Aut(F/\mathbb{Q})$.
   (c) The field $F \cap \mathbb{R}$.
   (d) All intermediate fields, identifying those which are Galois extensions of $\mathbb{Q}$.

   Hint: Find a polynomial for which F is a splitting field extension, and note that $\zeta + \zeta^{-1} = \alpha^2$.

(5) Let R be a commutative ring with 1. Define the terms *maximal ideal of* R, *prime ideal of* R, and *nilpotent element of* R.

   The *Jacobson radical* $J(R)$ is defined to be the intersection of the maximal ideals of R. Prove that:
   (a) $J(R)$ is an ideal of R.
   (b) For $a \in R$, $a \in J(R)$ if and only if $1 + ab$ is a unit for every $b \in R$.
   (c) Every nilpotent element of R is in $J(R)$.

(6) Let R be a commutative ring with 1, and define a *simple* R*-module* to be an R-module M such that $M \neq 0$ and the only submodules of M are 0 and M.
   (a) Let M be a simple R-module. Prove that M is *cyclic*, that is $M = Rm$ for some $m \in M$. Prove that M is isomorphic as an R-module to R/I where I is a maximal ideal of R. Hint: $r \mapsto rm$ is a surjective R-module homomorphism.
   (b) Let M and N both be simple R-modules and let $\alpha : M \to N$ be R-linear. Prove that either $\alpha = 0$ or α is an isomorphism between M and N.