ALGEBRA BASIC EXAM: JANUARY 2020

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be rings with 1, and all ring HMs are assumed to preserve 1.

(1) State and prove the Sylow theorem(s). Define the concept of *simple group*. Show that if p and q are distinct primes there is no simple group of order pq.

(2) Define the concepts of *solvable group*, *nilpotent group*, and *p-group* (for prime p). Prove that every nilpotent group is solvable and every finite p-group is nilpotent. Given an example of a solvable group which is not nilpotent.

(3) Define the terms *algebraic extension*, *separable extension*, *normal extension*, *splitting field*, *Galois extension*. State carefully and prove some version of the theorem that splitting fields exist. State carefully and prove some version of the theorem that splitting fields are unique up to isomorphism.

(4) State the Fundamental Theorem of Galois theory. Let F be the subfield of $\mathbb{C}$ generated by the roots of $x^4 - 2$. Determine with proof:
  (a) $[F : \mathbb{Q}]$.
  (b) The structure of $Aut(F/\mathbb{Q})$.
  (c) The field $F \cap \mathbb{R}$.
  (d) All intermediate fields, identifying those which are Galois extensions of $\mathbb{Q}$.

(5) Let R be a (not necessarily commutative) ring with 1. Define the *Jacobson radical* of R, and prove that $r \in R$ is in the Jacobson radical if and only if $1 + arb$ is a unit for all $a, b \in R$.

(6) Define the concepts of *principal ideal domain (PID)* and of a *finitely generated module* over a commutative ring. State and prove the structure theorem for finitely generated modules over a PID.