ALGEBRA BASIC EXAM: AUGUST 2019

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1, and all ring HMs are assumed to preserve 1.

(1) State the Sylow theorem(s). Prove that if G is a finite group and $N \lhd G$ then:
   (a) Every Sylow p-subgroup of N has the form $P \cap N$ for P some Sylow p-subgroup of G.
   (b) Every Sylow p-subgroup of G/N has the form PN/N for P some Sylow p-subgroup of G.

(2) Define the concepts of *solvable group*, *nilpotent group*, and *p-group* (for prime p). Prove that all finite p-groups are nilpotent.

(3) Define the concepts of *degree of an extension*, *normal extension*, *separable extension* and algebraic extension Let F_2 be a field extension of F_1, and let F_1 be a field extension of F_0.
   Prove or disprove (by means of an explicit counterexample):
   (a) If $[F_2 : F_1]$ and $[F_1 : F_0]$ are finite then $[F_2 : F_0]$ is finite.
   (b) If F_2/F_1 and F_1/F_0 are algebraic extensions then F_2/F_0 is algebraic.
   (c) If F_2/F_1 and F_1/F_0 are normal extensions then F_2/F_0 is normal.

(4) State the Fundamental Theorem of Galois theory. Let F/E be a separable extension of finite degree.
   Prove that:
   (a) Prove that there is an extension F'/F such that F'/E is a Galois extension of finite degree.
   (b) The extension F/E has finitely many intermediate fields.

(5) Define the terms *Noetherian ring*, *nilpotent element* and *nilradical*. Prove that if R is a Noetherian ring then there is a positive integer n such that $a^n = 0$ for all nilpotent elements a. Prove that this is not true for a general commutative ring with 1.

(6) State the structure theorem for finitely generated modules over a PID. Prove that if R is a PID and N is a torison-free finitely generated module then R is free. Show that this is false in general when R is a PID but N is not finitely generated.