ALGEBRA BASIC EXAM: JANUARY 2019

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1, and all ring HMs are assumed to preserve 1.

(1) State the Sylow theorem(s). Prove that if G is a finite group, $H \leq G$ and P is a Sylow p-subgroup of G, then there exists g such that $H \cap P^g$ is a Sylow p-subgroup of H.

(2) Define the concepts of *solvable group*, *nilpotent group*, and *p-group* (for prime p). Prove that if G is solvable then all subgroups and quotients of G are solvable. Prove that if $N \lhd G$ and the groups N and G/N are both solvable, then G is solvable.

(3) Define the terms *algebraic extension*, *separable extension*, *normal extension*, *splitting field*, *Galois extension*. Prove that an extension F/E of finite degree is normal if and only if it is a splitting field extension. You may use any form of the uniqueness of splitting field extensions so long as you state it clearly and correctly.

(4) State the Fundamental Theorem of Galois theory. Let F be the subfield of $\mathbb{C}$ generated by the roots of $x^6 - 3$. Determine with proof:
   (a) $[F : \mathbb{Q}]$.
   (b) The structure of $Aut(F/\mathbb{Q})$.
   (c) The field $F \cap \mathbb{R}$.
   (d) All intermediate fields, identifying those which are Galois extensions of $\mathbb{Q}$.

(5) Define the terms *prime ideal*, *maximal ideal*, *unit* and *nilpotent element*. Prove that if R is a ring and $x \in R$, then x is in every maximal ideal of R if and only if $1 + xy$ is a unit for every $y \in R$. You may use the fact that every proper ideal is contained in at least one maximal ideal.

(6) Define the terms *principal ideal domain* and *unique factorisation domain*. Prove that if R is a UFD then $R[x]$ is a UFD. You may use any form of Gauss' lemma so long as you state it clearly and correctly.