# Algebra Basic Exam, September 2018

Attempt **FOUR** of the following six problems; all have equal weight. Exactly one problem requires some form of the axiom of choice.

**Problem 1.** Prove Sylow's first theorem in the following form: if $G$ is a finite group and $p^k$ is a prime power dividing $|G|$, then $G$ possesses a subgroup of cardinality $p^k$.

**Problem 2.**

(a) Provide a definition of a *nilpotent group* in terms of the existence of a particular chain of subgroups (there are a couple of equivalent options; any will do).

(b) Prove that any finite group whose cardinality is a prime power is a nilpotent group.

**Problem 3.** Suppose that $R$ is a commutative ring with 1, and $r \in R$ is arbitrary. Show that $r$ is an element of every prime ideal of $R$ if and only if there is some $n \in \mathbb{N}$ with $r^n = 0$.

**Problem 4.** Suppose that $R$ is a commutative ring with 1 which is Noetherian. Show that its polynomial ring $R[x]$ is also a Noetherian ring.

**Problem 5.**

(a) Given a field $F$ and a polynomial $f \in F[x]$, define what it means for a field extension $K \supseteq F$ to be a *splitting field* of $f$ over $F$.

(b) Given a polynomial $f \in F[x]$ and two splitting fields $K_0, K_1$ of $f$ over $F$, prove that $K_0 \cong K_1$

**Problem 6.** Consider the polynomial $f = x^5 - 10x + 5 \in \mathbb{Q}[x]$, and let $K \subseteq \mathbb{C}$ be its splitting field over $\mathbb{Q}$. Identify $\mathrm{Aut}(K)$ with a familiar finite group, and prove that the groups are isomorphic. Here $\mathrm{Aut}(K)$ is the group of all automorphisms of $K$, equipped with the operation of composition.