ALGEBRA BASIC EXAM: SEPTEMBER 2017

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1, and all ring HMs are assumed to preserve 1.

(1) State and prove the Sylow theorem(s).

(2) Define the concepts of *solvable group*, *nilpotent group*, and *p-group* (for prime p). Prove that every finite p-group is nilpotent. Prove that if G is nilpotent and $H < G$ then $H < N_G(H)$.

(3) Define the terms *algebraic extension*, *separable extension*, *normal extension*, *splitting field*, *Galois extension*. State carefully and prove some version of the theorem that splitting fields are unique up to isomorphism.

(4) State the Fundamental Theorem of Galois theory. Let F be the subfield of $\mathbb{C}$ generated by the roots of $x^4 + 2$. Determine with proof:
   (a) $[F : \mathbb{Q}]$.
   (b) The structure of $Aut(F/\mathbb{Q})$.
   (c) The field $F \cap \mathbb{R}$.
   (d) All intermediate fields, identifying those which are Galois extensions of $\mathbb{Q}$.

(5) Define the terms *prime ideal*, *maximal ideal*, *unit* and *nilpotent element*. Prove that the sum of a unit and a nilpotent element is nilpotent. Prove that the nilpotent elements in a ring form an ideal, and prove further that this ideal is the intersection of all the prime ideals.

(6) State and prove the structure theorem for finitely generated modules over a PID.