ALGEBRA BASIC EXAM: JANUARY 2017

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1, and all ring HMs are assumed to preserve 1.

(1) State and prove the Sylow theorem(s).

(2) Define the terms *algebraic extension*, *separable extension*, *normal extension*, *splitting field*, *Galois extension*. Prove that if $[F : E] < \infty$ then F is a splitting field extension of E for some $f \in E[x]$ if and only if F is a normal extension of E. You may assume some version of the uniqueness of splitting field extensions provided that you state it clearly.

(3) Prove that if I is an ideal of R then R/I is a field if and only if I is maximal. Prove that for $a \in R$, a lies in some maximal ideal if and only if a is not a unit. Now let $R \neq 0$ and let $J(R)$ be the intersection of all the maximal ideals of R. Prove that $J(R) = \{a \in R : 1 + ab \text{ is a unit for all } b\}$.

(4) Let p be prime and let G be a finite group of order p^n for $n > 0$. Prove that $Z(G)$ is not the trivial subgroup. Prove further than $N \cap Z(G) \neq 1$ for all $N \triangleleft G$ with $N \neq 1$.

(5) State the Fundamental Theorem of Galois theory. Let F be the subfield of $\mathbb{C}$ generated by the roots of $x^6 - 1$. Determine with proof:
   (a) $[F : \mathbb{Q}]$.
   (b) The structure of $Aut(F/\mathbb{Q})$.
   (c) The field $F \cap \mathbb{R}$.
   (d) All intermediate fields, identifying those which are Galois extensions of $\mathbb{Q}$.

(6) Define the terms *Noetherian ring*, *finitely generated (fg) module*, *Noetherian module*. Prove that if R is Noetherian then every fg R-module is Noetherian, and show by a counterexample that this can fail when R is not assumed to be Noetherian.