ALGEBRA BASIC EXAM: JAN 2016

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1, and all ring HMs are assumed to preserve 1.

(1) State and prove the Sylow theorem(s).

(2) Define the terms *algebraic extension*, *separable extension*, *normal extension*, *splitting field*, *Galois extension*. Prove that if $[F : E]$ is finite then F is a normal extension of E if and only if F is a splitting field for some $f \in E[x]$. You may assume the result that splitting fields are unique, but must state it carefully and correctly for full credit.

(3) Define the terms *Noetherian ring* and *Noetherian module*. Prove that:
  (a) If $M \leq N$, then M and N/M are both Noetherian if and only if N is Noetherian.
  (b) if R is Noetherian and M is an fg R-module, then M is Noetherian.

(4) Define the concepts of *nilpotent group* and *solvable group*, and prove that nilpotent implies solvable. Prove that if a finite group has prime power order then it is nilpotent.

(5) State the Fundamental Theorem of Galois theory.
   Let F be the subfield of $\mathbb{C}$ which is generated by the roots of $x^3 + 2$. Determine (with proof) the Galois group of F over $\mathbb{Q}$, and find all the intermediate fields.

(6) Let α be a complex number which is algebraic over $\mathbb{Q}$. Prove that the following are equivalent:
  (a) α is the root of some monic polynomial in $\mathbb{Z}[x]$.
  (b) $m_\alpha^{\mathbb{Q}} \in \mathbb{Z}[x]$.