ALGEBRA BASIC EXAM: SEPTEMBER 2015

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1, and all ring HMs are assumed to preserve 1.

(1) State the Sylow theorems. Let G be a finite group, let $H \lhd G$ and let P be a Sylow p-subgroup of H. Prove that $G = HN_G(P)$.

(2) Define the terms *algebraic extension*, *separable extension*, *normal extension*, *splitting field*, *Galois extension*. Prove or disprove (by a counterexample) the following statement: if E_1 is a separable extension of E_0 and E_2 is a separable extension of E_1, then E_2 is a separable extension of E_0.

(3) Define the terms *Noetherian ring* and *Noetherian module*. Prove that if R is a Noetherian ring then
   (a) $R[x]$ is not a Noetherian R-module.
   (b) $R[x]$ is a Noetherian ring.

(4) Let p be prime and let G be a finite group of order p^n for $n > 0$. Prove that:
   (a) $Z(G)$ is not the trivial subgroup.
   (b) G has a normal subgroup of order p.
   (c) There exists an increasing chain $\langle N_i : 0 \leq i \leq n \rangle$ of normal subgroups of G with $|N_i| = p^i$.

(5) State the Fundamental Theorem of Galois theory.
   Let F be the subfield of $\mathbb{C}$ which is generated by the roots of $x^3 - 2$. Determine (with proof) the Galois group of F over $\mathbb{Q}$, and find all the intermediate fields.

(6) Let R be an integral domain, let F be the field of fractions of R and let P be a prime ideal of R. Let R/P be the quotient ring and let R_P be the subring of F consisting of fractions a/b with $b \notin P$.
   (a) Prove that the prime ideals of R/P are in 1-1 correspondence with prime ideals of R containing P.
   (b) Prove that the prime ideals of R_P are in 1-1 correspondence with prime ideals of R contained in P.