ALGEBRA BASIC EXAM: JANUARY 2015

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1.

(1) Let the group G act on the set X. Denote the stabiliser of $x \in X$ by G_x, the orbit of x by O_x, and the set $\{x \in X : g \cdot x = x\}$ by $Fix(g)$.
  (a) State and prove the orbit-stabiliser relation.
  (b) Now assume that G and X are finite.
    (i) Prove that $\sum_{g\in G} |Fix(g)| = \sum_{x\in X} |G_x|$.
    (ii) Prove that the number of orbits is
$$\frac{\sum_{g\in G} |Fix(g)|}{|G|}.$$

(2) State Sylow's theorem(s).
  Show that in a group of order 48, the intersection of two distinct Sylow 2-subgroups has order 8.

(3) Define the terms *separable extension*, *normal extension*, *splitting field*, *Galois extension.*
  Let $[F : E] = n$ and let $\sigma : E \to E'$ be a monomorphism from E to an algebraically closed field E'. Prove that the following are equivalent:
  (a) F is a separable extension of E.
  (b) There are exactly n monomorphisms $\tau : F \to E$ such that $\tau \restriction E = \sigma$.

(4) State the Fundamental Theorem of Galois theory.
  Let F be the subfield of $\mathbb{C}$ which is generated by the roots of $x^4 + 2$. Determine (with proof) the Galois group of F over $\mathbb{Q}$, and find all the intermediate fields.

(5) Prove that if R is a PID then every fg R-module is a direct sum of finitely many cyclic R-modules. Give an example (with proof) of a PID R and an R-module which is not expressible as a direct sum of finitely many cyclic R-modules.

(6) Prove that every PID is a UFD.
  Let k be a field and let R be the polynomial ring $k[x]$. Show that the quotient of the polynomial ring $R[y]$ by the principal ideal generated by $1 - xy$ is a PID.