ALGEBRA BASIC EXAM: SEPTEMBER 2014

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1.

(1) (a) Let the group G act on the set X. Define the terms "orbit" and "stabiliser". State and prove the orbit-stabiliser relation.
(b) Let p be prime and let G be a group of order p^n for some $n > 0$. By considering the action of G on G by conjugation, or otherwise, prove that $Z(G) \neq 1$.
(c) Let p be prime and let G be a group of order p^n for some $n > 0$. Prove that $[G, G] < G$ where $[G, G]$ is the commutator subgroup of G. Hint: Use the last part to power an induction.

(2) Let R be a ring and let M be an R-module. Define $M[x]$ to be the set of polynomials with coefficients in M, and make $M[x]$ into an $R[x]$-module in the obvious way. Prove that if M is a Noetherian R-module then $M[x]$ is a Noetherian $R[x]$-module.

(3) State the Fundamental Theorem of Galois theory. Let F be the unique subfield of $\mathbb{C}$ which is a splitting field for $x^4 - 2$ over $\mathbb{Q}$. Identify (with proof) the fields intermediate between $\mathbb{Q}$ and F which are Galois extensions of $\mathbb{Q}$.

(4) Define the concepts "Noetherian ring", "principal ideal domain", "Euclidean domain", "unique factorisation domain", "prime element", "irreducible element". Let R be an ID. Prove that:
(a) If R is a PID, R is Noetherian.
(b) Every prime element is irreducible.
(c) If R is a PID then every irreducible element is prime.
(d) If R is Noetherian and every irreducible element is prime, then R is a UFD.

(5) State and prove the Jordan-Hölder Theorem.

(6) Define the terms *separable extension*, *normal extension*, *splitting field*, *Galois extension*. Prove or disprove by a counterexample the following statement: if E_1 is a normal extension of E_0 and E_2 is a normal extension of E_1, then E_2 is a normal extension of E_0.