Basic Examination Sample

Measure and Integration

Solve three of the following problems.

1. State and prove Egoroff's theorem.

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $\mathcal{L}^1(E) > 0$. Prove that for every $0 < t < \mathcal{L}^1(E)$ there exists a Lebesgue measurable subset $F \subset E$ such that $\mathcal{L}^1(F) = t$.

3. Consider the function

$$F(y) = \int_0^\infty \frac{e^{-yx}}{1+x^2}\, dx, \quad y \geq 0.$$

   (a) Prove that F is continuous.

   (b) Prove that F is differentiable for $y > 0$.

   (c) Prove that F' is differentiable for $y > 0$.

   (d) Prove that $F''(y) + F(y) = \frac{1}{y}$ for all $y > 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that there exists $M \geq 0$ such that

$$|f'(x)| \leq M$$

for all $x \in [a, b]$ for some $a < b$.

   (a) Prove that f' is Borel measurable.

   (b) Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f'(x)\, dx,$$

   where $f_n(x) := n\left[f\left(x + \frac{1}{n}\right) - f(x)\right]$, $x \in \mathbb{R}$.

   (c) Prove that

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Justify your work.