Basic Examination Sample

Measure and Integration

Solve three of the following problems.

1. Let $(X, \mathfrak{M}, \mu)$ be a measure space. State and prove Hölder's inequality in $L^p(X)$, $1 \leq p \leq \infty$.

2. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f \in L^1(X) \cap L^\infty(X)$.

   (a) Prove that f belongs to $L^p(X)$ for all $1 < p < \infty$.

   (b) Prove that if μ is finite, then for all $1 < p < \infty$,
   $$\|f\|_{L^p} \leq \|f\|_{L^\infty} (\mu(X))^{1/p}.$$

   (c) Prove that if μ is finite, then for any sequence $\{p_n\}$ of numbers satisfying $p_n > 1$ and $p_n \to \infty$ as $n \to \infty$,
   $$\limsup_{n\to\infty} \|f\|_{L^{p_n}} \leq \|f\|_{L^\infty}.$$

   (d) Prove that if μ is finite, then
   $$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}.$$

3. Let $E \subset \mathbb{R}^N$ be bounded and define the *Lebesgue inner measure* of E as
   $$\mathcal{L}_i^N(E) := \sup\left\{\mathcal{L}_o^N(C) : C \text{ closed } C \subset E\right\}.$$

   (a) Prove that E is Lebesgue measurable if and only if $\mathcal{L}_i^N(E) = \mathcal{L}_o^N(E)$.

   (b) Prove that E is Lebesgue measurable if and only if there exists an F_σ set F and a G_δ set G with $F \subset E \subset G$ such that $\mathcal{L}^N(G \setminus E) = 0$ (hence the σ-algebra of Lebesgue measurable sets is the completion of the Borel σ-algebra).

4. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that
   $$\lim_{x\to\infty} f(x) = \ell \in \mathbb{R}.$$
   Prove that for every $a > 0$,
   $$\lim_{n\to\infty} \int_0^a f(nx)\, dx = a\ell.$$

Justify your work.