Discrete Mathematics and Probabilistic Combinatorics (4 Hours)

Attempt TWO problems from each section.

Discrete Mathematics

Problem 1 i) State the Infinite Ramsey Theorem.

ii) By adopting the proof of the Infinite Ramsey Theorem or otherwise, prove the following statement.

Let c be a coloring of pairs of $\mathbb{N} = \{1, 2, \ldots\}$ such that for every $x \in \mathbb{N}$ the pairs $\{x, y\}$ with $y \in \mathbb{N} \setminus \{x\}$ receive only finitely many colors. (The total number of colors may be infinite.) Then there is an infinite set $Y \subseteq \mathbb{N}$ such that either all pairs of Y receive the same color or for every $u, v, x, y \in Y$ with $u < v$ and $x < y$ we have $c(\{u, v\}) = c(\{x, y\})$ if and only if $x = u$.

Problem 2 i) Carefully state the general (2-variable) version of the Exponential Formula.

ii) Let $s(n, k)$ be the Stirling number of the first kind, that is, the number of permutations of $\{1, \ldots, n\}$ whose cycle decomposition consists of exactly k cycles. Each fixed point is counted as a separate cycle. Also, we agree that $s(0, 0) = 1$. Find the closed form for

$$S(y, z) = \sum_{n,k\geq 0} s(n, k)\, y^k\, \frac{z^n}{n!}.$$

iii) A *friendship graph* is a union of cliques which share a common vertex but are otherwise vertex-disjoint. For example, if we have cliques of orders $n_1, \ldots, n_k$, then the corresponding friendship graph has $1 + \sum_{i=1}^{k}(n_i - 1)$ vertices and $\sum_{i=1}^{k} \binom{n_i}{2}$ edges. Find the exponential generating function for f_n, the number of friendship graphs with vertex set $\{1, \ldots, n\}$. (Your answer may involve integrals.)

Problem 3 i) Show that every graph G with average degree $\bar{d}(G)$ least 99 contains a non-empty subgraph with minimum degree at least 50. Also, demonstrate that we cannot replace 50 by 51 in the above statement, that is, construct a graph G with $\bar{d}(G) \geq 99$ such that every non-empty $H \subseteq G$ has minimum degree at most 50.

ii) Let G be the graph on 3-subsets of $\{1, \ldots, n\}$ in which A and B are adjacent if and only if $|A \cap B|$ is even. Assuming only the standard facts of linear algebra, prove that neither G nor its complement $\overline{G}$ contains K_{n+1}, the complete graph of order $n + 1$.

Probabilistic Combinatorics

Problem 4 i) State Markov's inequality.

ii) Let $\mathrm{ext}(G)$ be the largest k such that G has the *k-extension property* (that is, for any disjoint $A, B \subseteq V(G)$ with $|A \cup B| \leq k$ there is a vertex $x \in V(G) \setminus (A \cup B)$ which is connected to everything in A but to nothing in B). Prove that for the random graph $G_{n,1/2}$ we have with probability $1 - o(1)$ that

$$\mathrm{ext}(G) = (1 + o(1)) \log_2 n.$$

iii) Prove that the vertices of every k-uniform hypergraph with less than 2^{k-1} edges can be colored with two colors so that no edge is monochromatic.

Problem 5 i) State and prove Chebyshev's inequality.

ii) Let $G_{n,p}$ be the random graph with edge probability p and let the random variable X count the number of isolated edges in $G_{n,p}$. Let $n \to \infty$ and suppose that $p = p(n)$ is a function of n such that the expectation of X tends to infinity. Prove that $X \geq 100$ with probability $1 - o(1)$.

Problem 6 i) State the Lovász Local Lemma (both versions). Show how the general version implies the symmetric one.

ii) Let a graph G have maximum degree $d \geq 1$ and let $V_1, \ldots, V_r \subseteq V(G)$ be pairwise disjoint sets, each of cardinality $\lceil 2ed \rceil$. Prove that G contains a stable set W which intersects every V_i. (A set $W \subseteq V(G)$ is *stable* if it spans no edge in G.)