Department of Mathematical Sciences
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Sample Basic Examination
Differential Equations

Time allowed: 3 hours.

Problem 1

Let $A \in \mathbb{R}^{n\times n}$ and suppose that its spectrum,

$$\sigma(A) = \{z \in \mathbb{C} \mid \det(A - zI_{n\times n}) = 0\} \subset \mathbb{C},$$

is such that

$$\sigma(A) \subset \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}.$$

Let $B : [0,\infty) \to \mathbb{R}^{n\times n}$ be a continuous matrix-valued function such that

$$\int_0^\infty \|B(t)\|_2 \, dt < \infty,$$

where here we have written $\|M\|_2 = \left(\sum_{i,j=1}^n |M_{ij}|^2\right)^{1/2}$ for any $M \in \mathbb{R}^{n\times n}$. Prove that 0 is an asymptotically stable solution to the ODE

$$\dot{x}(t) = (A + B(t))x(t).$$

Problem 2

Let Γ denote the fundamental solution to the Laplacian (a.k.a. the Newton potential).

1. State the formula for Γ in $\mathbb{R}^n$ for $n \geq 2$. Define any constants that you use.
2. Suppose that $\varphi \in C_c^2(\mathbb{R}^n)$, i.e. φ is twice differentiable and has compact support. Prove that the function $u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\varphi(y)dy$$

is well-defined and that $u \in C^2(\mathbb{R}^n)$ and satisfies $\Delta u = \varphi$.

Problem 3

Suppose that $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a bounded solution to the heat equation:

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) & \text{for } x \in \mathbb{R}, t > 0 \\ u(\cdot, 0) = f \end{cases}$$

for $f \in C_b^0(\mathbb{R}) = C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$\lim_{x \to -\infty} f(x) = a \text{ and } \lim_{x \to \infty} f(x) = b.$$

Show that $\lim_{t \to \infty} u(x,t)$ exists for all $x \in \mathbb{R}$ and compute its value.

Problem 4

Consider the problem

$$\begin{cases} \partial_t^2 u(x,t) = \partial_x^2 u(x,t) & \text{for } x, t > 0, \\ u(x,0) = g(x) & \text{for } x > 0 \\ \partial_t u(x,0) = h(x) & \text{for } x > 0 \\ \partial_t u(0,t) + \lambda \partial_x u(0,t) = 0 & \text{for } t > 0, \end{cases}$$

for initial data $g, h \in C^\infty([0,\infty))$ satisfying $g(0) = h(0) = 0$.

1. Assume that $\lambda \neq 1$. Give an explicit formula for the solution u. [Hint: You may consider the general solution to the wave equation: $F(x+t) + G(x-t)$.]
2. Now assume that $\lambda = 1$. Present data g and h for which the problem has no solution. Prove that there is no solution.

Problem 5

Find the entropy solution to the following problem:

$$\begin{cases} \partial_t u(x,t) + (u(x,t))^3 \partial_x u(x,t) = 0 & \text{for } x \in \mathbb{R}, t > 0 \\ u(\cdot, 0) = g, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0. \end{cases}$$