Department of Mathematical Sciences
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# Sample Basic Examination
# Differential Equations

**Time allowed: 3 hours.**

## Problem 1

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous, i.e. there exists a constant $K \geq 0$ such that

$$|f(x) - f(y)| \leq K\,|x - y| \text{ for all } x, y \in \mathbb{R}^n.$$

1. Prove that the initial value problem

$$\text{(IVP)}: \quad \begin{cases} \dot{x}(t) = f(x(t)) & \text{for } t \geq 0 \\ x(0) = x_0 \end{cases}$$

   admits a unique global solution (i.e. the solution exists for all $t \geq 0$) for every choice of initial data $x_0 \in \mathbb{R}^n$.

2. Define the flow map associated to $f$ to be the map $\eta : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ given by $\eta(y, t) = x(t)$, where $x$ solves (IVP) with $x_0 = y$. Prove that for each $t \geq 0$, the map $\eta(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous.

3. Let $\eta$ be the flow map associated to $f$. Prove that for $t, s \geq 0$ and $y \in \mathbb{R}^n$ we have the identity

$$\eta(\eta(y, s), t) = \eta(y, t + s).$$

## Problem 2

1. Let $u$ be a positive harmonic function on a ball $B(0, 2r) \subset \mathbb{R}^n$ for $r > 0$. Show that for all $x \in B(0, r)$

$$r^{n-2}\,\frac{r - |x|}{(r + |x|)^{n-1}}\,u(0) \leq u(x) \leq r^{n-2}\,\frac{r + |x|}{(r - |x|)^{n-1}}\,u(0).$$

   Hint: Use the Poisson's formula: Let $w$ be a harmonic function on ball $B(0, R)$ equal to a continuous function $g$ on $\partial B(0, R)$. Then

$$w(z) = \frac{R^2 - |z|^2}{n\,\mathrm{Vol}(B(0,1))\,R} \int_{\partial B(0,R)} \frac{g(y)}{|z - y|^n} dS_y.$$

2. Recall that Liouville's theorem establishes that every bounded harmonic function on $\mathbb{R}^n$ must be constant. Show that in fact every positive harmonic function on $\mathbb{R}^n$ must be constant.

## Problem 3

Suppose that $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is the solution to

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(\cdot, 0) = g \end{cases}$$

for some $g : \mathbb{R}^n \to \mathbb{R}$ that is bounded and continuous. Further suppose that there exist constants $A, a > 0$ such that we have the bound $|u(x,t)| \leq Ae^{a|x|^2}$ for all $x \in \mathbb{R}^n$ and $t \geq 0$.

1. Let $R \in O(n) = \{M \in \mathbb{R}^{n\times n} \mid MM^T = I\}$. We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is $R-$invariant if $f(Rx) = f(x)$ for all $x \in \mathbb{R}^n$. Prove that if $g$ is $R-$invariant, then $u(\cdot, t)$ is $R-$invariant for each $t \in [0, \infty)$.

2. Prove that if $g$ is radial, then $u(\cdot, t)$ is radial for each $t \in [0, \infty)$.

3. Let $\omega \in \mathbb{R}^n \backslash \{0\}$. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is $\omega-$periodic if $f(x + \omega) = f(x)$ for all $x \in \mathbb{R}^n$. Prove that if $g$ is $\omega-$periodic, then $u(\cdot, t)$ is $\omega-$periodic for each $t \in [0, \infty)$.

4. Prove that if $g$ is odd / even then $u(\cdot, t)$ is odd / even for each $t \in [0, \infty)$.

## Problem 4

Suppose that $g, h \in C_c^\infty(\mathbb{R})$. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ be the solution to the wave equation

$$\begin{cases} \partial_t^2 u(x,t) = \partial_x^2 u(x,t) & \text{for } x \in \mathbb{R}, t > 0 \\ u(\cdot, 0) = g, \partial_t u(\cdot, 0) = h. \end{cases}$$

1. Prove that
$$\int_{\mathbb{R}} \left[ |\partial_t u(x,t)|^2 + |\partial_x u(x,t)|^2 \right] dx = \int_{\mathbb{R}} \left[ |h(x)|^2 + |\partial_x g(x)|^2 \right] dx$$
for all $t > 0$.

2. Prove that there exists $T > 0$ such that for $t \geq T$ we have the "equipartition of energy" identity
$$\int_{\mathbb{R}} |\partial_x u(x,t)|^2 \, dx = \int_{\mathbb{R}} |\partial_t u(x,t)|^2 \, dx.$$

## Problem 5

Suppose that $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is continuous. Solve the transport equation

$$\begin{cases} \partial_t u(x,t) + a(x,t) \cdot \nabla u(x,t) = f(x,t) & \text{for } x \in \mathbb{R}^n, t > 0 \\ u(x, 0) = g(x) \end{cases}$$

for the following choices of $a : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$.

1. $a(x,t) = Ax + b$ for $A$ a constant $n \times n$ matrix and $b \in \mathbb{R}^n$ a constant.

2. $a(x,t) = bh(t)$ for $b \in \mathbb{R}^n$ a constant and $h : \mathbb{R} \to \mathbb{R}$ continuous.

3. $n = 1$ and $a(x,t) = -tx$.