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Measures of Simmelian Tie Strength, Simmelian Brokerage, and, the Simmelianly Brokered

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Abstract

A set of measures for Simmelian tie strength, Simmelian brokerage, and, being Simmelianly brokered are introduced. The measures are derived from interpretations of a quote from Simmel (1950). The theoretically most informative measure of Simmelian brokerage is based on a complex value measure of Simmelian tie strength reflected in an Hermitian matrix. Also measures based on weight matrices and hypergraphs are discussed. A maximum for the number of ties one node could Simmelian broker in a network of n nodes is determined.

1 Introduction

Simmel's sociological theory (Simmel, 1950) has tremendous influence on modern social network theory. For example, the basic ideas of structural hole theory are inspired by Simmel's ideas on brokers (Burt, 1992). However, also work related to group norm behavior can be traced back to Simmel (Krackhardt, 1998). One intriguing idea first presented by Krackhardt (1999) about brokers of cliques has prompted this paper. He defines Simmelian ties as ties embedded in cliques. However, he doesn't give a measure of Simmelian brokerage. Furthermore, some measures of Simmelian ties and brokerage have been suggested and used in the literature. However, derivation of those measures are often ad hoc. Usually no direct connection is made to Simmel's work in the development of the measures.

In this paper a set of Simmelian tie and broker measures is defined that are directly related to one quote from (Simmel, 1950). Four interpretations of this quote are made explicit. Based on each interpretation a measure or set of measures is defined. These measures of Simmelian tie strength are used to determine Simmelian brokerage. Furthermore, the paper derives extremal values for the tie as well as the brokerage measures.

2 Theoretical Background

Krackhardt (1998) defines a Simmelian tie as a tie embedded in a clique, because Simmel (1950) argues that group-size doesn't fundamentally change the impact of groups on behavior, rather the change from dyad to triad or larger groups changes individuals' behavior. Simmel (1950, p.138) states:

> "Dyads thus have very specific features.This is shown not only by the fact that the addition of a third person completely changes them, but also, and even more so, by the common observation that the further expansion to four or more by no means correspondingly modifies the group any further".

The change occurs mainly because in groups of three or larger, group norms become an effective means of coordination (see also, Coleman, 1990). Therefore, belonging to a group is more predictive of behavior than the size of the group, according to Simmel (1950).

Krackhardt (1998) illustrates that Simmelian ties are very strong and have a longer longevity than non-Simmelian ties. These ties have at least three consequences for its constituents that Krackhardt (1999) describes as *reduced individuality*, *reduced bargaining power*, and, *enhanced conflict resolution*. The main social mechanism in Simmelian ties that affects individuals is norm consistent behavior.

This leads Krackhardt (1999) to bring up the issue of brokerage of Simmelian ties. In case an individual is member of different cliques she will face different sets of role expectations. According to role theory this will evoke role stress for these individuals (see Krackhardt, 1999; Merton, 1968; Kahn et al., 1964). Simmelian brokerage hence is harmful to individuals' performance.

Unintentionally, Burt (1998) shows that Simmelian brokerage could have positive effects, especially for members of minority groups. In his paper on the gender of social capital he suggests that individuals that belong to a minority group benefit from a broker that belong to a social majority. In the mechanism that Burt (1998) describes the broker assures the "validity" of the individual from the minority group, and acts as a mentor. An implicit assumption is that the broker introduces the "protege" to other contacts. This implies that the mentor makes the protege more structurally equivalent, i.e. also a broker. However, when the mentor and protege share ties with a number of mutually unconnected individuals these ties become Simmelian (assuming symmetric ties). The structural equivalence between mentor and protege implies that both become Simmelian brokers (see figure 1).

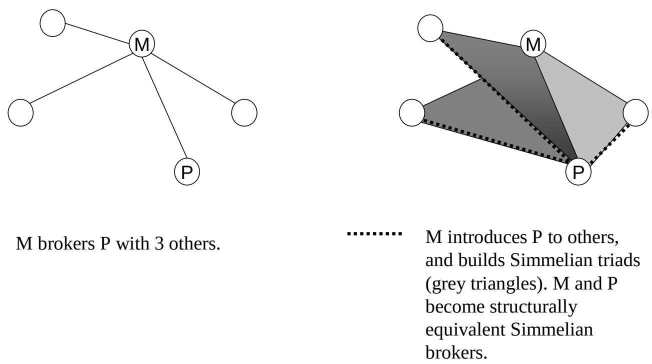


Figure 1: Simmelian brokerage through mentoring.

When Simmelian brokerage as shared social capital (Burt, 1998) is beneficial, it could hurt those that are being brokered. In fact, here it is proffered that when individual h faces a number of Simmelian brokers, this could be a source for role stress. Therefore we will not only discuss, measures of Simmelian tie strength and Simmelian brokers, but also a measure to indicate the extend to which someone is Simmelianly brokered.

To derive some theoretical conditions for Simmelian tie and brokerage measures from Simmel (1950, p.138) we need to realize that there are at least four possible different interpretations of his sentence on the effects of adding a third to a dyad. Each of these interpretations leads to different measures. When considering the behavioral opportunities and restrictions that a Simmelian tie implies for an individual,

1. the size of a group adds no further insight; a dichotomous measure is sufficient.
2. the size of a group adds no further insight, rather the number of groups are more important, because these groups are the sources

of behavioral restrictions (see Krackhardt, 1999); a measure that indicates the number of groups two individuals share is sufficient.

3. the impact on behavioral restrictions of group size increases at a decreasing rate. The strength of behavioral restrictions depends on the number of others in joint groups; a measure that indicates the number of individuals with whom two individuals share a group is sufficient.
4. both interpretations 2 and 3 apply; a measure that combines sources and strength of behavioral restrictions is sufficient.

Which interpretation is right most likely depends on application and research context. In the next section several measures for Simmelian ties are derived, which are each consistent with one of the interpretation stated above.

3 Simmelian Tie Measures

Interpretation 1 above suggests that a dichotomous measure of Simmelian ties is sufficient. Now, let Z be an $n \times n$ binary matrix representing the presence (Z_{hj}=1) and absence (Z_{hj}=0) of relationships between n individuals. Also, let Y be $Z \cap Z'$ representing the symmetric ties in Z. Now, a measure of Simmelian ties consistent with the definition in (Krackhardt, 1998, 1999) and Simmel's statement is

$$S = Y \otimes (Y^2) \tag{1}$$

The operator $\otimes$ indicates the element-wise Boolean operation that produces 1 if both terms > 0. So, a Simmelian tie between i and j exists whenever there is a symmetric tie between h and j and they each have a symmetric tie in common with at least one other node. S in equation (1) reflects the presence $S_{hj} = 1$ or absence $S_{hj} = 0$ of a Simmelian tie. This result is identical to the result of the hypergraph approach that Krackhardt and Kilduff (2002) propose. However equation (1) presents a less computationally intensive approach.

Interpretation 2 above suggests that a measure that indicates the number of groups two individuals share is sufficient. In fact, Krackhardt (1999) derives just such a measure. Krackhardt (1999) uses interpretation 2 to formally define Simmelian ties as symmetric relationships embedded in a clique. He assumes that the number of individuals in the clique is immaterial to the effect that a clique has on one member's behavior.

Cliques are formally defined as the maximum groups of individuals in which each individual has a symmetric relationships with each other member of the group (Luce and Perry, 1949). Symmetric relationships exist if and only if there is a relationship from h to j, and a relationship from j to h. The phrase 'maximum group' implies that we cannot add any other individual to the group that has symmetric relationships with *all* other members of the group.

Hence, based on the Luce and Perry (1949) algorithm Krackhardt (1999) defines the clique matrix as

$$C_{hk} = \begin{cases} 1 & \text{if an individual } h \text{ belongs to clique } k \\ 0 & \text{if an individual } h \text{ does not belong to clique } k \end{cases} \tag{2}$$

To identify Simmelian ties Krackhardt (1999) calculates the co-clique matrix as

$$K = CC', \tag{3}$$

where C' is the transpose of C. Note that the diagonal of K indicates the number of cliques to which each individual is a member. The off-diagonal values indicate the number of cliques two individuals share. This could be interpreted as the strength of a Simmelian tie in terms of the number of sources for behavioral constraint.

Interpretation 3 suggests that a measure that indicates the number of individuals with whom two individuals share a group is sufficient. Such a measure respects the assertion Simmel makes, when we assume that further expansion of a clique to four or more has some effect. Different aspects of group-based behavior might in fact immediately depend on the number of others in that group; e.g., Friedkin (2001) on models for opinion formation.

In that case we have

$$SQ = Y \cdot (Y^2) \tag{4}$$

where [·] is the element-wise multiplication operator. This measure indicates the number of individuals q, $q \in k$ that have symmetric ties with both h and j. These individuals q could be distributed over one or different cliques. This means that h and j could share multiple sets of norms when they both are members of a number of cliques. It could be reasoned that irrespective of the number of cliques h and j share (see equation 3), the number of individuals with whom they share clique(s) (SQ_{hj}) might indicate how effective the norms between h and j are. For example, an important sociological mechanism is gossip where the individuals q are central in opinion formation about behavior and intentions of both h and j (cf., Merry 1984; Coleman 1990). The relationships between h and q, and, j and q impute the constraints on h and j, because they observe behavior of both, check its consistency to norms, and tell others (h or j and other q) about inconsistencies. In this reasoning the number of individuals q enhances the constraints h and j put on each others behavior, because more of their behaviors become observable to each other.

Interpretation 4 suggests we should take information from measures that take both the strength of behavioral constraint (interpretation 3) as well as the number of sources for behavioral constraint (interpretation 2) into account. There are several ways to include information on both in one measure. Three ways will be considered here. First, one could consider constructing some kind of weight matrix. Second, based on equations (3) and (4) a Hermitian matrix can be constructed, which is a square self-adjoint complex number matrix [1]. We derive from the Hermitian martix a real value measure for relative Simmelian tie strength, which is consistent with interpretation 4. Furthermore, in section 5.4 on how we can use eigenvalues to determine Simmelian brokerage the Hermitian matrix approach proves to be valuable. Third we explore some possibilities of using hypergraphs.

3.1 Weight Matrices

Let us define a weighting matrix of dimension $k \times k$,

$$W = \omega(diag(C'C)) \tag{5}$$

where W, the $k \times k$ diagonal weighting matrix, is a function of the diagonal of $C'C$ that represent the number of individuals in clique k. For example, if we wish to assume that clique size linearly increases constraint we would use $\omega(diag(C'C))$ to define the diagonal values

$$W_1 = (diag(C'C))^{-1} \tag{6}$$

This weighting matrix represents the share each individual has in a clique. A large share indicates a small clique, while a small share indicates a large clique. The maximal value is 1/3 as the minimal clique size is 3; the minimal value is $1/n$ when there is one clique to which all nodes in the Simmelian graph belong. However, the weights in equation (6) are independent, and hence say nothing about relative strength of behavioral restrictions in comparison to other cliques.

A specification that takes relative strength into consideration is

$$W_2 = \frac{1}{Tr(diag(C'C))} diag(C'C) \tag{7}$$

[1] A complex number has the form $a + bi$, where i denotes the imaginary number $\sqrt{-1}$, a is said to be the real part of the complex number, and b is said to be the imaginary part of the complex number. The conjugate of $a + bi$ is $a - bi$. Self-adjoint means that taking the conjugate transpose of matrix H returns the same matrix H. Self-adjoint matrices are Hermitian matrices.

where $diag(C'C)$ is a diagonal matrix with dimensions $k \times k$ that represents the number of individuals per clique, $Tr(diag(C'C))$ is the trace of that matrix representing the total sum of individuals in all cliques. If individuals reside in more than one clique, than it is possible that $Tr(diag(C'C)) > n$. Note that the diagonal of W_2 in equation (7) sums to unity. The maximum value possible in W_2 equals 1 when there is only one clique. When there is more than 1 clique, the maximal value decreases. The minimal value is $3/Tr(diag(C'C))$ for the minimum clique.

When there are many cliques, irrespective of their relative size, the weights in W_2 become small. This is potentially problematic, because the weight matrix is intended to capture the strength of restrictions from a clique independent of the number of sources of restrictions (number of cliques). A measure that is robust against the number of cliques is

$$W_3 = \frac{1}{\max(diag(C'C))} diag(C'C) \tag{8}$$

where $\max(diag(C'C))$ is the size of the largest clique. It attains a minimum of $3/(n-1)$, which implies there are two cliques, one of which contains 3 nodes and the largest one that contains $n-1$ nodes. This means two nodes in the large clique are connected to a node that has no Simmelian ties than with those two. It attains a maximum of 1 for the largest clique irrespective of the number of cliques.

Thus, equation (6) weights the importance of individuals within cliques; equation (7) is a weight for the importance of a clique relative to all other cliques based on the number of individuals; equation (8) measures the relative size of a clique compared to the largest clique. Note however, that $W_2 \sim W_3$, because they only differ in the specification of a constant that weights $diag(C'C)$. The relevance of the different specifications of the weighting matrix follows from the context in which the measures are applied.

Consider,

$$K_\omega = CWC'. \tag{9}$$

If W is the identity matrix of dimensions $k \times k$, equation (9) implies $K_\omega = K$. Hence, equation (3), Krackhardt's (1999) measure of Simmelian ties is a special case of the weighted Simmelian tie measure.

Now, we could use the specification in equation (6) to get

$$K_{W_1} = CW_1C'. \tag{10}$$

The hj^{th}-element of equation (10) sums the inverse of sizes of cliques that contain h and j as members. Only, when the size of a clique is assumed to have a meaningful interpretation irrespective of other cliques' sizes equation (10) is a useful measure.

As absolute clique-size often is less informative than relative clique-size researchers might prefer using the specification in equation (7) to get

$$K_{W_2} = CW_2C' \tag{11}$$

which shows the extend to which two nodes, h and j, have their Simmelian ties based in large cliques. The maximum value in K_{W_2} is 1, when h and j belong to all cliques. This immediately shows a disadvantage of this measure, because the maximum value does not reflect any information on size.

Alternatively, the specification in equation (8) is less dependent on the number of cliques. This measure gives

$$K_{W_3} = CW_3C', \tag{12}$$

which shows the sum of the weighted joint cliques, where the weight is the size relative to the largest clique. The sum is equal to the number of cliques h and j share, when the sizes of all cliques are the same, or when they share only the largest clique. Smaller values occur when they share smaller cliques.

Equations (10), (11), and, (12) are examples of weighted Simmelian tie matrices that allow to incorporate the strength of behavioral restrictions of Simmelian ties dependent on the cliques in which they are embedded. Many other weight matrices could be specified.

3.2 Hermitian Matrices

Let us assume that K in (3) and SQ in (4) both hold relevant information about the Simmelian state of a social network, i.e. interpretation 4 is preferred. Hoser and Geyer-Schulz (2005) suggest to use complex numbers to represent two aspects of a social network (more specifically, inties and outties). Social network matrices, like K and SQ that each reflect one facet of the social structure, can similarly be expressed as one complex matrix, more specifically as an Hermitian matrix. Such an Hermitian matrix preserves the information available in both matrices, and has some attractive properties for determining Simmelian brokerage, such as real eigenvalues (see section 5.4). To construct a Hermitian matrix, first let

$$M = M_{hj} = \begin{cases} K_{hj} + SQ_{hj}\, i & \forall\, h < j \\ SQ_{hj} + K_{hj}\, i & \forall\, h > j \\ 0 & \text{if } h = j \end{cases} \tag{13}$$

be a the complex matrix where by construction in the upper diagonal the values of K constitute the real part and the values of SQ reflect the imaginary part, while in the lower diagonal it is vice versa. Note also that K and SQ are always symmetric, and therefore $M_{hj} = i\bar{M}_{jh}$, where $\bar{M}$ indicates the complex conjugate of M. Now, Hoser and Geyer-Schulz (2005) show that multiplying equation (13) with $e^{-i\frac{\pi}{4}}$ produces an Hermitian matrix:

$$H = M\ e^{-i\frac{\pi}{4}}. \tag{14}$$

Although the values in equation (14) are complex and hence cannot be rank ordered, H contains all the information that is present in K and SQ. Hoser and Geyer-Schulz (2005, p.274) discuss properties of a Hermitian matrix based on in- and outties, which immediately translate to the setting of Simmelian ties (equation 14). Recall, that SQ_{hj} indicates the number of individuals q, $q \ \epsilon\ k$ that have symmetric ties with both h and j, while K_{hj} indicates the number of cliques that h and j share. Now if h and j share less cliques than there are individuals with whom they both have symmetric ties, $K_{hj} < SQ_{hj}$, it follows from Hoser and Geyer-Schulz (2005, p.274) that the imaginary part of H_{hj} has a positive sign, while the imaginary part of H_{jh} has a negative sign. Furthermore, the imaginary part of H_{hj} is 0 if $K_{hj} = SQ_{hj}$.

Note, that

$$K_{hj} \leq SQ_{hj} \tag{15}$$

in Simmelian networks, because adding a minimal clique to the number of cliques h and j share, increases the number of individuals q also with one. Adding a clique larger than a minimal clique increases the number of individuals q even more, while the number of cliques increases by one. This inequality is very useful to derive a relative measure of Simmelian tie strength, to indicate among how many cliques the Simmelian tie between j and h is scattered relative to the number of individuals q that constitute this Simmelian tie. In other words, a measure that indicates the number of cliques of which the tie between j and h is a part, relative to the number of indirect (two step) ties between j and h.

This measure follows from the exponential form of complex numbers. It is well known that complex numbers can be expressed in several manners, for example

$$a \pm bi = Re^{\pm i\theta} \tag{16}$$

where $R = (a^2 + b^2)^{\frac{1}{2}}$ is the absolute value of the complex number. Furthermore,

$$\begin{aligned} \sin(\theta) &= b/R \\ \cos(\theta) &= a/R \end{aligned} \tag{17}$$

which we can solve for θ in the interval $[0, 2\pi)$, where θ is called the *phase*. The phases of the complex numbers in H will be called the *Simmelian phases*.

Applying equality (16) to M_{hj}, we get

$$R_{hj} = (K_{hj}^2 + SQ_{hj}^2)^{\frac{1}{2}} \tag{18}$$

, and

$$\begin{aligned} \sin(\theta) &= SQ_{hj}/R_{hj} \\ \cos(\theta) &= K_{hj}/R_{hj} \end{aligned} \tag{19}$$

Now, if $K_{hj} = SQ_{hj}$, i.e., every q with a Simmelian tie to h and j constitutes a minimal clique, than

$$R_{hj} = K_{hj}\sqrt{2} = SQ_{hj}\sqrt{2} \tag{20}$$

and,

$$\sin(\theta) = \cos(\theta) = SQ_{hj}/R_{hj} = K_{hj}/R_{hj} = \frac{1}{2}\sqrt{2} \tag{21}$$

which implies that in this case, $\theta = \frac{1}{4}\pi$. This is a minimum value, because when $SQ_{hj} \to \infty$, while $K_{hj} = 1$, i.e. h and j are members of one, but very large clique, we have

$$\begin{aligned} \sin(\theta_{hj}) = & \quad SQ_{hj}/R_{hj} \to 1 \\ \cos(\theta_{hj}) = & \quad K_{hj}/R_{hj} \to 0 \end{aligned} \tag{22}$$

which implies $\theta_{hj} \to \frac{1}{2}\pi$. Hence, the interval for θ is $[\frac{1}{4}\pi, \frac{1}{2}\pi)$ if we take M_{hj} as basis. If we use H_{hj} than the interval for $\theta_{H_{hj}}$, given equation (14), becomes $[0, \frac{1}{4}\pi)$. Note that the intervals for M_{jh} and H_{jh} shift, although the range remains $\frac{1}{4}\pi$.

If the value of the Simmelian phase associated with the tie between h and j becomes smaller, than the dependence of h on j to maintain joint norms in the cliques they are both members of increases. In fact, the phase could be interpretted as a measure of joint Simmelian brokerage or in terms of the relative degree of borrowed social capital if we assume *joint* Simmelian brokerage is a source of benefits (cf. Burt, 1998). Hence, the Simmelian phase matrix (P) could be a relative measure of tie strength. This matrix is skew-symmetric. Here it is convenient to construct a symmetric matrix P, such that $P = P_{hj} = \theta_{H_{hj}} = P_{jh}$.

Hermitian matrix expression of Simmelian ties has other advantages. The eigensystem of H is informative about the state of Simmelian brokerage in a social network. In the section on Simmelian brokerage, we use the eigenvalues of H to determine the strength of Simmelian brokerage.

3.3 Hypergraphs

A third method to capture both aspects of Simmelian ties (number of cliques and number of third parties that constitute the Simmelian tie) proceeds along the way Krackhardt and Kilduff (2002) have suggested. As mentioned above, they use hypergraphs to determine the number of individuals that support a Simmelian tie (see for more information on hypergraphs Berge, 1989).

Similar to Krackhardt and Kilduff (2002), we could use a hypergraph to determine the number of cliques two individuals share. If we distill from the hypergraph HG_1, a $n \times k$ clique-membership matrix comparable to K. Furthermore, as complete triads constitute Simmelian ties also a matrix HG_2, $n \times t$ "complete triad"-member matrix could be derived comparable to SQ.

This type of matrix is especially useful when the triad is unit of analysis, for example in analysis of bargaining power and dominant coalitions. For example, $CT = HG_2' \otimes HG_1$, a "complete triad" by clique matrix, which could be used to determine how often a specific complete triad is a part of different cliques. Furthermore, $T = CT\, CT'$, would give a valued complete triad by complete triad matrix, that indicates how often two Simmelian triads are constituents of the same cliques. Standard measures of influence could than be used to determine the influence of a complete triad in a network of triads. Relevant, for our analysis here is that we could use the matrix T in analysis of Simmelian brokerage we discuss below. Additionally, this type of reasoning could be applied to Simmelian dyads, as well as larger sized clique sub-groups, such as complete tetrads, complete pentads, etc. However, in the analysis of larger clique subgroups the smaller cliques are ignored, and strictly speaking information on the Simmelian state of the social network is lost.

4 Valued Raw Ties

We have not considered valued relations, because the Simmelian tie theory Krackhardt (1999) refers to is a nominal theory of group influence. In fact, the strong cohesive groups to which the Simmelian tie theory refers implies strong relationships between individuals. Weak relationships are of less importance and classified together with the non-existent ties. At least as far as definitions of cliques are used to operationalize this theory.

Therefore the definition of Simmelian ties (Krackhardt, 1998) forces researchers to decide on a cut-off value (c)(cf. Doreian, 1969) to determine what tie strength in the raw data warrants the minimal strength of a Simmelian tie ($S_{hj}Z_{hj} = S^v_{hj} > c$). This decision implies establishing the minimum tie strength between two individuals that warrants the existence of a group according to Simmel's sociology.

However, if a research focus moves from strong cohesive groups to brokers of strong cohesive groups it becomes very relevant to assess weaker ties. As Granovetter (1973) suggest broker relationships are on average weaker than non-broker relationships. Hence, tie strength becomes a valuable piece of information for our analysis.

Another reason to consider tie strength in measuring Simmelian broker positions is that Krackhardt's (1999) paper positions itself as an refinement of Burt's (1992) structural hole theory. This theory allows for valued ties. Although, the paper of Krackhardt surely qualifies to be an independent new theory comparison to Burt's original theory remains important. This requires explicit consideration of how to handle valued ties.

Consideration of tie strength is especially important when we take interpretation 3 and 4, and use the proposed measures that are based on the number of individuals q that constitute a Simmelian tie between h and j. In fact the value of the Simmelian tie between h and j is a function of the value of ties between h and j, h and q, j and q. Also one could argue that the value of the ties among the q individuals matter.

Assuming that raw tie strength affects Simmelian tie strength we have to consider how to use such information in our measures. A straightforward way to take tie value into account is to take the raw tie value as Simmelian tie strength (when a tie is indeed Simmelian). However, what if we have asymmetric raw data. Furthermore, we need to consider the fact that the Simmelian nature of ties is dependent on indirect ties. Hence, how to deal with indirect tie strength is a second issue.

The first issue pertains when $S^v_{hj} \neq S^v_{jh}$. Theoretically one could argue that a Simmelian tie is as strong as it's weakest tie and choose $S^v_{hj} = \min(Z_{hj}, Z_{jh})$ as Simmelian tie strength. Although, individual researchers may want to select own decision criteria appropriate for their specific study setting.

The second issue offers even more discretionary choice to researchers. Not only do decisions on whether the values of the constituent ties between h and q, and j and q need to be incorporated, also decisions on the functional form are required. For example, does one use an additive, multiplicative or other specification. Substantive interpretation of the measures should be a guide in choosing one specific measure.

5 Simmelian Brokerage Measures

Brokerage refers to an individual network position that presents opportunities for extraordinary behavior, but also incurs costs, because an actor is a link between others that are not directly connected. For example, cut-points in a network are brokers, individuals with bridging ties are brokers, but these are all sufficient conditions. A broker position is due to the fact that an individual is connected to two or more others who do not share a direct tie. As Burt (1992) emphasizes, a broker fills a hole between two or more others. Several measures are available to measure the extent to which someone is in a broker position.

On the other hand it may useful to know how often a person is Simmelianly brokered. Imagine when you face a majority clique between you and another person (see figure 2). While unconnected with that other person, the clique may extort a formidable force on both to connect. In other words the stress the clique experiences because the split between you and that other person forces them into two cliques, may be put back to you. First, we discuss characteristics of Simmelian brokers, next we discuss characteristics of the Simmelianly brokered.

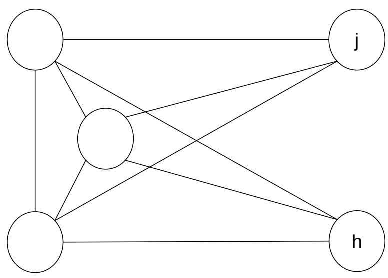


Figure 2: Nodes h and j both face a majority clique. The hole between h and j divides the group in two cliques.

5.1 Characteristics of the Simmelian Broker

Given a specific Simmelian tie measure the question is how we can determine whether h brokers the Simmelian tie with j. In other words, does h has a Simmelian tie with node q who has no Simmelian tie with j. The procedure Krackhardt (1999) proposes doesn't allow us to derive the degree to which individuals hold a Simmelian broker position. However, given S, it is straightforward to determine the number of Simmelian ties that h brokers for j, i.e. how many others, q, are there for which, $S_{hq} = 1$, and $S_{jq} = 0$.

Let S^c be the complement of S (S^c= I^c-S, where I^c is the complement of the $n \times n$ identity matrix I). As before, let [$\cdot$] be the element-wise multiplication operator. Now, whether h brokers j is given by,

$$SB = S \otimes SS^c, \tag{23}$$

while the number of Simmelian ties h brokers for j is

$$SBQ = S \cdot SS^c. \tag{24}$$

Equation (24) counts all the nodes that h has Simmelian ties with excluding j, and with whom j has no Simmelian ties. In other words, equation (24) measures the difference between the total number of individuals with whom h has Simmelian ties, and the number of individuals with whom h and j both have a Simmelian tie.

Introducing the valued Simmelian ties in this measure can be done in different ways. Let us define SV as a valued Simmelian tie. Now a very straightforward way is to take as value of Simmelian brokerage the value of the tie it self. We would get

$$SVB_0 = SV \cdot SB. \tag{25}$$

To indicate the intensity of behavioral restrictions from one tie that need to be harmonized with other sets of behavioral restrictions. If it is required that the Simmelian broker measure indicates the tension between the sets of restrictions another measure is needed. We would need to include the value of the $h \longleftrightarrow j$ tie as well as the value of the $h \longleftrightarrow q$ ties. The following equation expresses the product of the value of tie $h \longleftrightarrow j$ with the sum of the values of ties $h \longleftrightarrow q$ when $j \longleftrightarrow q = 0$.

$$SVB_1 = SV \cdot SVS^c \tag{26}$$

An alternative of the product of the valued ties that h brokers for j is the sum of ties $h \longleftrightarrow j$ and $h \longleftrightarrow q$. This is given by

$$SVB_2 = SV \cdot SB + SVS^c. \tag{27}$$

Equations (26) and (27) are indicators of the total amount of restrictions h has to cope with between j and all other q for $S_{jq} = 0$. A third measure of interest might be the difference,

$$SVB_3 = SV \cdot SB - SVS^c, \tag{28}$$

which indicates whether the impact of the restrictions j imposes are stronger or weaker than the average of the restrictions that the q nodes impose ($S_{jq} = 0$).

These measures are a function of the number of nodes in a network. To make the measures comparable over networks we formulate relative measures based on the maximum value that these measures can take in a network with n nodes.

It is trivial to determine the maximum tie value in the dichotomous matrices in equations (1) and (23), which is 1. In equation (4) that measures Simmelian tie strength as the number of indirect ties between h and j that constitute their Simmelian tie the maximum tie value is $n-2$. This maximum will be attained if all possible indirect ties are realized in a one clique network. In that case there will be no Simmelian broker in the network.

The maximum brokering occurs if h is a broker between j and all possible others. A Simmelian tie between h and j implies that at least one other is connected to both these nodes. Therefore the degree of j, $d(j)$, has a minimum when $d(j) = \delta(j) = 2$. Consequently the maximum number of ties that h can broker for j in a network with n nodes is $n-3$, if the degree of h, has a maximum when $d(i) = \Delta(i) = n-1$.

It is very important to note that this maximum can not be attained for the ties with every j, rather this is dependent on network size. In networks that contain an even number of nodes there will be an odd number of nodes with ties to h, and at least one of them will have to have ties with two others. Thus, the maximum number of ties that h could possibly broker for j in a Simmelian network of n individuals is not automatically the same as the average number of ties h brokers when she is a maximal broker for all nodes.

More formally, the handshaking lemma states that the sum of degrees in a graph is twice the number of edges. Consequently, every graph has an even number of nodes of odd degree. When $\Delta(h)$ (maximum degree) is odd this implies that there must be a j with odd degree. As the minimum degree in a Simmelian graph is $\delta(k) = 2$ there must be a j with degree $d(j) = 3$ if h is a maximal broker for the other $n-2$ nodes. In the case where $d(i) = n-1$, h can broker only $n-4$ ties for this particular j that has $d(j) = 3$, while the other q nodes have $d(q) = 2$.

So, when n is odd ($n-1$ is even) h could broker $n-3$ between all $n-1$ other nodes, which is $\frac{1}{2}[(n-1)(n-3)]$; when n is even ($n-1$ is odd) h could broker $n-3$ between $n-2$ nodes, and between $n-4$ and the 1 node that has $d(j) = 3$. Hence, the maximum number of Simmelian ties that h can broker in a network of n nodes is

$$\max \sum_{j} SBQ_{hj} = \lceil \frac{1}{2}[(n-1)(n-3)-1] \rceil \tag{29}$$

where $\lceil . \rceil$ is the ceiling function that rounds the result of the expression up to nearest integer when the decimal is larger than zero.

5.2 Characteristics of the Simmelianly Brokered

As discussed above, when j faces a (majority) clique that brokers her with q, one could argue that she experiences more pressure to bond with q. Hence, being Simmelianly brokered might also be a source of role stress. The maximum number of Simmelian brokers that j could face is $n - 2$, i.e. all other nodes are brokers between j and q. The number of times j and q are brokered is given by

$$SD = S^2 \cdot S^c \tag{30}$$

However, given interpretation 2 it might be more valuable to determine how many nodes q broker h with clique l. This is given by,

$$F_1 = (C'S)' \cdot C^c \tag{31}$$

where K^c is the complement of the clique-member matrix. Hence, F_1 is a 'clique-non-member' matrix ($i \times k$), where the non-members have at least a Simmelian tie to one clique member.

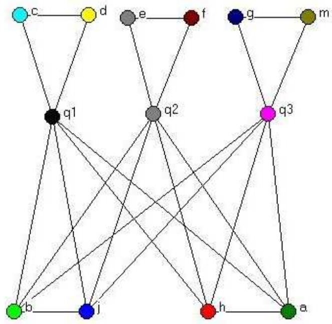
However, equation (31) does not tell us how many cliques of which h is not a member and j is a member contain at least one node q that brokers between h and j (note that we don't want to identify all individual q's nor count the number of q's). This is given by

$$F_2 = F_1[C'SB \otimes C'] \cdot S^c \tag{32}$$

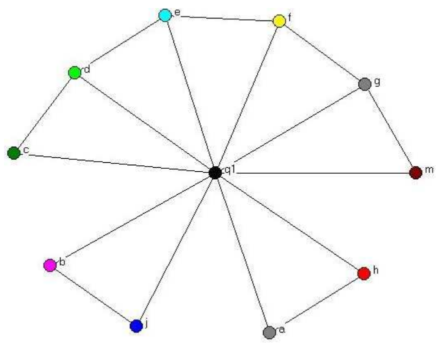
The first part of equation (32) indicates whether there are nodes in clique l that have a tie with node h, while h is no member of clique l. The second part of equation (32) states whether there are nodes in clique l that broker j, where j is a member of clique l. The third part ensures there is no Simmelian tie between h and j. Now F_2 is a measure that counts how many cliques of which j is a member, also contain the Simmelian broker(s) of h and j. Note that this measure is dependent on the symmetry of Simmelian ties.

The measure in equation (32) quantifies the amount of pressure on h to close the hole with j, because of the divers set of cliques j and the Simmelian broker('s) of h share. If j shares more than one clique with them, she is to them, at least to some extent, a structural equivalent Simmelian broker. Assuming that the tie between j and the Simmelian broker('s) is beneficial to them (cf., Burt, 1998) it implies that although j and h have no direct tie j could be a source of stress to h.

In some instances it could be relevant to count the number of cliques that include Simmelian brokers q, but neither h nor j. For example, when we want to assess the pressure on h and j to close the hole between them. If the broker(s) q belong(s) to many cliques that neither



(a) The two pairs *h*-*a* and *j*-*b* both Simmelianly broker three others, who are involved in three separate cliques. In turn, these three Simmelianly broker the two pairs. Incentives for *h* and *j* to close the hole between them could include reducing role stress, resisting pressurization to adopt different norms or to reduce the bargaining power of the three.



(b) The pairs *h*-*a* and *j*-*b* face one Simmelianly broker, q_1, involved in many different cliques. In all other cliques q_1 has a brokering partner that helps sustain more flexible norms for q_1 and herself. As the pairs have more to lose when severing their ties to q_1 than vice versa, there are incentives for *h* and *j* to establish a connection.

Figure 3: Examples of Simmelian broker structures that provide incentives for the Simmelianly brokered to change structure.

h nor j belong to this evokes an incentive for h and j to close the hole between them (see figure 3). This measure is given by

$$F_3 = F_1[C'SB \otimes C^{c'}] \odot S^c \tag{33}$$

Equation (33) differs from equation (32) in the second part, where $C^{c'}$ (the complement of the clique-membership matrix transposed) replaces C'. Thus, instead of summing all the cliques that include both j and q, equation (33) sums all the cliques that contain q but not j.

If we define F_4 as the sum of all cliques that contain brokers between h and j we have

$$F_4 = F_2 + F_3 \tag{34}$$

Similar to our complex valued measure of Simmelian tie strength, we can derive a measure of Simmelianly brokered tie strength

$$M_2 = M_{2_{hj}} = \begin{cases} F_{t_{hj}} + SD_{hj}\, i & \forall\, h < j \\ SD_{hj} + F_{t_{hj}}\, i & \forall\, h > j \\ 0 & \text{if } h = j \end{cases} \tag{35}$$

where $t = 2, 3, 4$. However, the properties of M_2 will be different from M (equation 13). Note that the number of cliques a broker belongs to is independent of the number of brokers between h and j. Hence, we can not derive a restriction like inequality (15), and thus no boundaries on M_2 in equation (35).

5.3 Contstraint

The rationale behind Burt's (1992) measure of brokerage is to identify the *lack of* brokerage opportunities. He argues that such lack of entrepreneurial opportunities constraints individual behavior. However, this is not necessarily Simmelian constraint. In his constraint measure, Burt (1992) discards information about a-symmetries between h and j, as he uses the sum of the tie from h to j and from j to h. Therefore, only when we apply constraint to a Simmelian network it might be used as a measure of *lack of* Simmelian brokerage.

When we want to analyze the measure of constraint we must be alert to a few aspects that follow from the fact that Burt (1992) develops a measure tailored for ego-network data. This has some important implications when we want to apply this measure to complete networks. First, we can only apply the constraint measure to 'single component networks', i.e. we can only apply it to weakly connected graphs. All individuals must be at least weakly reachable. Burt (1992) normalizes the data on the basis of the sum of inties and outties of individual h, which is the focal individual (or ego). Hence normalization of data occurs on the basis of individual specific values. For isolates constraint of individual is undetermined.

One advantage of using constraint is that it solves the second issue concerning raw valued data by implicitly deciding on how to deal with indirect values. However, as the other measures discussed above constraint assumes real numbers. So we can not use the Hermitian matrices of Simmelian ties. Of course we could use the θ-matrix, P, as it contains only real numbers.

5.4 Eigenspectral Analysis

The Simmelian tie measures appropriate to determine Simmelian brokerage in the measures discussed above did not include the Hermitian matrices. However, this Simmelian tie measure includes most information on the Simmelian ties, and hence could be most informative about the advantages and restrictions posed by brokering these ties. One measure of brokerage, similar to closeness centrality (see Freeman, 1979), is eigenvector centrality (see Bonacich, 1972, 1987).

Let,

$$\lambda \, g = SV \, g \tag{36}$$

where λ is the largest eigenvalue, and g is the associated eigenvector. Now, based on Bonacich (1987) we can define Bonacich-centrality as

$$c(\alpha, \beta) = \alpha \, (I - \beta \, SV)^{-1} SV 1 \tag{37}$$

where 1 is a column vector of ones and I is an identity matrix. Furthermore, we know that if $\beta \rightarrow 1/\lambda$ that $g \rightarrow c(\alpha, \beta)$ (Bonacich, 1987). One interpretation of $c(\alpha, \beta)$ that Bonacich (1987) provides is that it equals closeness centrality of Freeman (1979), in that it increases when h is connected to others through shorter paths.

Unfortunately, this does not say anything about Simmelian brokerage, because in a complete Simmelian network every node would have the shortest path to every other node. Therefore, the values in g should be normalized to the minimum value in g. After normalization values larger than one imply that node h is a broker, as long as we assume that SV is a one component network. Otherwise, a measure for each component could be derived, although comparability between nodes in different components is no longer valid.

In principle we could let SV in equation (36) be H, an Hermitian matrix. As the eigenvalues, λ_H, obtained from H are real, $1/\lambda_H$ exists. However, the eigenvector components may be complex, which begs a further explanation about how to interpret the eigenvector values.

Hoser and Geyer-Schulz (2005, p.272) suggest that the largest absolute eigenvector component of the eigenvector associated to the largest absolute eigenvalue indicates the most active group member. If we apply this to H then, the most active group member is the one with the strongest Simmelian ties, in terms of both the number of cliques as well as the number of Simmelian ties. The absolute value of the other eigenvector components reflect the extend to which other group members have Simmelian ties with the most active member.

Whether the most active member is a Simmelian broker cannot be inferred from the first eigenvector. However, Hoser and Geyer-Schulz (2005) show that when an individual has the highest eigenvector component value in a second eigenvector this indicates that the individual is the center in a star-like structure.

In a Simmelian network this implies a Simmelian broker. Assessment of whether individuals hold Simmelian broker positions and to what extend, first requires an ordering of dominance of substructures on the basis of absolute eigenvalues. Second, it requires comparison over different eigenvectors of relative absolute eigenvector component values (relative with respect to the absolute eigenvector component values from the same eigenvector). A highest relative absolute eigenvector component value in two eigenvectors implies Simmelian brokerage.

Furthermore, the phases of eigenvector component values tell us something about the extend to which others that are in contact with the most central individual, are structural equivalent Simmelian brokers. Assume that h is the most central Simmelian broker. Whenever the cliques h that j share are minimal cliques, thus $K_{hj} = SQ_{hj}$, the phase of the eigenvector component equals zero of $-\pi$ Hoser and Geyer-Schulz (see, 2005). Now if $K_{hj} > 1$ this subsequently implies that values of the phase closer to 0 and $-\pi$ indicate that j is at least partly a structural equivalent Simmelian broker to h.

6 Conclusion

In this paper many different ways to measure Simmelian tie strength, Simmelian brokerage and Simmelianly brokered ties have been presented. This set of measures could be useful to test Simmel's theory or theories that were derived from his work (e.g., Krackhardt, 1999). Notice that these measures were based on four different interpretations of one quote of Simmel. The main advantage of the Simmelian tie strength measures is that they are firmly rooted in a substantive theory. However, this also limits the use of the measures. Researchers are warned that applying these measures out-side the scope of Simmel's or derived theories, begs a solid interpretation.

Based on Hermitian matrices a relative measure of Simmelian tie strength was developed, which includes information about both the number of cliques two individuals share, as well as the number of constituents of their Simmelian tie. In combination with measures that look at only one facet of Simmelian ties this measure allows to get a more fine-grained insight in network structural effects on for example individual perceptions, attitudes, behavior. The paper shows how these different tie strength measures can be used to determine to what extent one individual Simmelianly brokers the other. This is important when we want to empirically assess Simmelian theories.

Furthermore, the paper suggests some measures to identify the extent to which individuals are brokered. In the network literature this has received not much attention. Most work focusses on brokerage or non-brokerage (e.g., Burt, 2000). However, the counter-effects of brokerage by those who are brokered, in our case Simmelian brokered are usually not discussed. Although, perhaps less problematic in cross-sectional research, it seems very important when we look at network dynamics. The value of all measures discussed here needs to be shown in empirical research. Further theoretical developments are needed with respect to the eigenvalue decomposition of Hermitian Simmelian matrices.

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