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Eigenvector Centrality: Illustrations Supporting the Utility of Extracting More Than One Eigenvector to Obtain Additional Insights into Networks and Interdependent Structures

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Abstract

Among the many centrality indices used to detect structures of actors' positions in networks is the use of the first eigenvector of an adjacency matrix that captures the connections among the actors. This research considers the seeming pervasive current practice of using only the first eigenvector. It is shows that, as in other statistical applications of eigenvectors, subsequent vectors can also contain illuminating information. Several small examples, and Freeman's EIES network, are used to illustrate that while the first eigenvector is certainly informative, the second (and subsequent) eigenvector(s) can also be equally tractable and informative.

Keywords: centrality, eigenvector centrality, social networks

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Scholars who study social networks often begin with an analysis to determine which actors are the most important, or the most central to a network. For example, in an office environment, to understand a network of colleagues, it would be important to identify the most important players in that office environment. A challenge in social network analysis comes in trying to figure out the best way to understand importance or centrality. It could be that the actors with the largest number of ties are the most important (as reflected by degree centrality), for example the colleague who has meetings with the largest number of other colleagues, yet presumably quantity does not equal quality, and frequent meetings can be a waste of time. On the other hand, it could be that the colleagues who seem to be the link between other colleagues are important or powerful, given that their connections imply a means of access among others in the network (reflected by betweenness centrality). However, neither of these measures would take into account the simple fact that there is more power in being connected to powerful people than there is in being connected to a lot of people with limited access or resources. Eigenvector centrality is a centrality index that calculates the centrality of an actor based not only on their connections, but also based on the centrality of that actor's connections.

Thus, eigenvector centrality can be important, and furthermore, social networks and their study are more popular than ever. Eigenvector centralities have become a staple centrality index, along with degree, closeness, and betweenness (recall: degrees reflect volumes and strengths of ties, closeness captures the extent to which relations traverse few "degrees of separation," and betweenness highlights actors who connect sections of the network; Freeman, 1979). All four centrality indices are included in social network texts (cf., Knoke and Yang, 2007; Scott, 2012; Wasserman and Faust, 1994), and in research articles that compare the performance of centrality indices (cf. Borgatti 2005; Borgatti, Carley, and Krackhardt, 2006; Costenbader and Valente, 2003; Friedkin, 1991; Rothenberg et al., 1995; Smith and Moody, 2013; Stephenson and Zelen, 1989), as well as in the major social network analysis software packages (cf., UCINet, Pajek, NetMiner, NetworkX and LibSNA, NodeXL and SNAP, even Mathematica and StatNet).

When using eigenvector-based centrality, early definitions and current practice are focused on the first eigenvector of the sociomatrix that contains the ties among the actors. The reasoning is sound in that the first eigenvector is associated with the largest eigenvalue, thus capturing the majority of the variance contained in the network. However, there often remains further information about the network structure that subsequent eigenvectors can explain. For example, where the first eigenvector is likely to reflect volumes and strengths of connections among the actors, a second or third eigenvector can delineate those in separate groups within the network who behave in somewhat equivalent manners, or other elements of network structure that can be informative in understanding the actors and the patterns that link them. The research in this paper is conducted to demonstrate that the extraction of only the first eigenvector can be, and in even modest-sized networks typically will be, insufficient for a more comprehensive understanding of the network.

This research is not intended to produce a new centrality measure; rather to evaluate the status of the eigenvector centrality, and suggest that extending it beyond the extraction of only the first eigenvector can be insightful, as illustrated with several examples. To this end, this paper demonstrates that network scholars who consider additional eigenvectors (second, third, and subsequent) will typically be rewarded in obtaining richer insights about additional aspects of network interdependencies. Even that recommendation might not be said to cover "new ground" in that early social network scholars (e.g., Comrey, 1962) seem to have been more willing to consider multiple eigenvectors, such as research modeling networks of “consensus analysis” in anthropology (e.g., Romney, Weller, and Batchelder, 1986; see also Kumbasar, 1996). However, more recent practice has slipped back toward a simpler reduction of deriving only a single eigenvector, and to not consider the greater vector portfolio would seem to be a lost opportunity. A reviewer also noted that this issue may be all the more relevant in today’s scholarship, given the relevance of eigenvector, or eigenvector-like structures, in different models and domains. For example, much of “community detection” regularly relies upon singular values (recall these are like eigenvectors, but drawn from asymmetric matrices; Wang and Sukthankar, 2015). In addition, most latent space models are ultimately based on eigenvector-like structures, including some of the recent work on exponential random graphs (cf., Hoff, Raftery, and Handcock, 2002; Hoff and Ward, 2004).

The remainder of this paper is organized as follows:

1. Eigenvector centrality is reviewed—its conceptual and mathematical definition.
2. Several simple networks are used to illustrate that a single eigenvector may indeed suffice to characterize the network, but that with very little additional complications in structure, very often driven by sheer size, a second or third eigenvector (or more) will be helpful and informative in describing additional aspects of the actors’ positions in a network.
3. The eigenvectors are then analyzed for a known, real social network, the electronic exchanges in the EIES Freeman data (Freeman and Freeman, 1979). It is shown that the first eigenvector is correlated with (i.e., somewhat redundant with) other standard measures of centrality, and the second eigenvector illuminates other structural properties in the network that are shown to be related to information on the actors’ attributes.

The paper concludes by suggesting that network scholars may wish to modify how they proceed with eigenvector centralities, treating them more analogously to traditional uses of eigenvalues and eigenvectors, such as in principal components, namely by extracting multiple vectors.

Eigenvectors—Basics and Centrality

Before turning to eigenvector centralities eigenvectors are first briefly reviewed. The essence of the questions underlying eigenvector-based analyses share the quest for data reduction, from some number of raw variables to a smaller set of vectors, such as principal components or factors, that somehow capture or approximate reasonably well the variability or information in the raw data. Sometimes a single vector will suffice, but frequently more eigenvectors are needed, and one central question in principal components and factor analysis is: How many components or factors to extract?

More precisely, eigenvalues and eigenvectors form the basis of multivariate statistical models, such as principal components and factor analysis (e.g., Kim and Mueller, 1978; Manly, 1986; see Appendix A). In those models, researchers pose the question as to whether a set of p variables might share sufficient covariability to be described by a single, underlying principal component or factor. For example, it might be the case that a person’s subjective ratings of his or her “perceived health” and “mobility” may be both adequately described by age (or perceived age), with the concept (or “factor”) of age serving as the underlying principle construct,, meaning knowledge of the person’s age would be sufficient for estimations of the person’s likely standings on perceived health and mobility.

The mechanics of the model are familiar: a dataset $\boldsymbol{X}$ that is N (sample size) by p (number of variables) is processed into a $p\times p$ correlation matrix, $\boldsymbol{R}$. The correlation matrix is factored into two unique matrices: one of eigenvalues $\boldsymbol{\Lambda}$ (ordered $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$), one of eigenvectors $\boldsymbol{V}$ (each column with entries $v_1,\ v_2, \ldots\ v_p$), and its transpose $\boldsymbol{V}'$, $\boldsymbol{R} = \boldsymbol{V\Lambda V}'$, such that $y_i = v_1 X_{i1} + v_2 X_{i2} + \cdots + v_p X_{ip}$ (Seber, 1984).

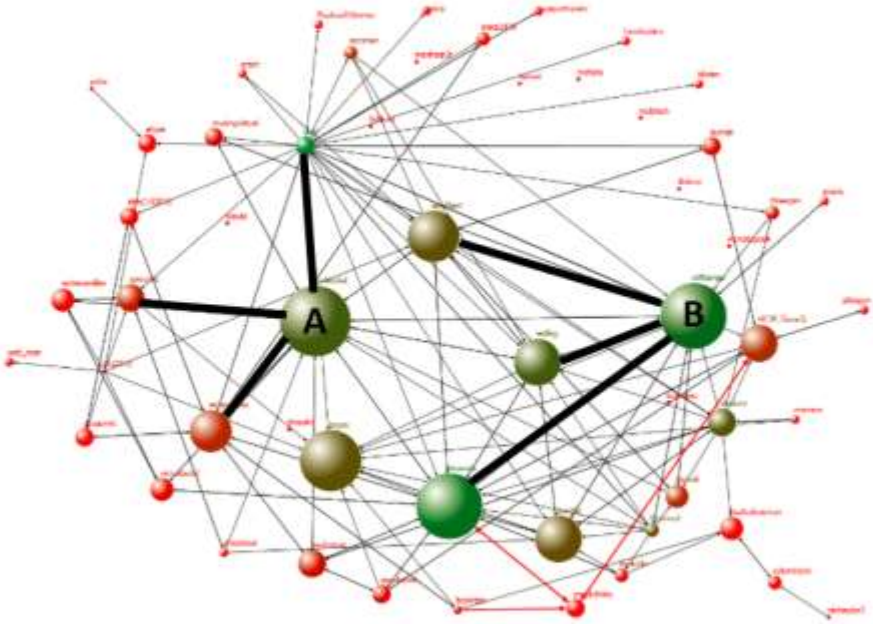
The first eigenvector (column of $\boldsymbol{V}$) contains the weights that will optimally transform the original p variables (e.g., health and mobility) into a single new score (e.g., a scale of perceived age) that explains the maximum possible variance in the data matrix $\boldsymbol{X}$ (that variance being λ). The first eigenvector yields weights for each variable. If the weight coefficients are high for health and mobility, then instead of working with both health and mobility as separate variables in subsequent analyses, it should be acceptable to use the perceived age scale instead (i.e. doing so would be sufficient in explaining the health and mobility data, and it may be optimal in terms of parsimony to use one rather than two variables, per this example). Subsequent eigenvectors 2, 3, …, p will contain weights that create new variables that explain the maximum amount of remaining variance (such as smoking history, which would not be explained by health, mobility, or age) subject to the constraint that each newly created variable is uncorrelated (not redundant) with the previous composite variables (Tabachnick and Fidell, 2006).

The application of eigen-models is not typically limited to the extraction of only the first eigenvector (Kim and Mueller, 1978; Seber, 1984). In principal components, the frequently employed heuristic is to extract as many eigenvectors as there exist eigenvalues that exceed 1.0. The reasoning is that given that the eigenvalue is the variance of the composite score formed using the weights in the eigenvector, the new composite score should explain at least as much variance as that in a single variable, which is 1.0 as expressed in standardized form, such as in a correlation matrix (Manly, 1986). In factor analysis, the eigenvalues are examined for their relative size, and the number of factors is determined to be that which corresponds to the number of relatively large eigenvalues (Tabachnick and Fidell, 2006).[1] A sociomatrix is not a correlation or covariance matrix, so the rule of thumb to extract as many eigenvectors as there are eigenvalues that exceed 1.0 (as the new composite variable's unit variance as in principal components analysis) is not directly applicable. Instead, the judgment of the relative size of the ordered eigenvalues (as in factor analysis) is the rule of thumb that transfers more readily in the application to social networks. And of course, it would be prudent to not extract eigenvectors associated with eigenvalues that are zero or negative.

In sum, many research articles in the social and physical sciences find it useful to extract more than one eigenvector—the amount and patterns of variability in the source data warrant doing so. Some data may certainly be analyzed and captured sufficiently with a single component or factor, but it seems that many more papers report multiple components or factors due to the complexities of the data and the research questions at hand.

[1] Note that these analyses are obviously conducted on square, symmetric correlation matrices. When an eigenvector model is used on a square, symmetric sociomatrix, it will operate similarly. If the sociomatrix is not symmetric, or if it is two-mode, a singular value decomposition yields analogous information (Namboodiri, 1984; Seber, 1984).

Figure 1. Real Social Network that would Yield Different Actor Eigenvector Centralities*



*This figure is public domain, obtained via an image search on Google of the key word "network," with the option of "usage rights" set to "free to use or share, even commercially." http://www.smrfoundation.org/2009/09/27/social-media-network-analysis-workshop-october-29th-in-mountain-view-ca/.

The finding in this paper will suggest that this truism should carry over in the analysis of social network data as well. That is, sometimes a network might be explained thoroughly with a single eigenvector (as seen through examples after the general review of eigenvector centralities). However, what seems largely unexplored in the analysis of social networks is how much more clearly and comprehensively the structure of a network may be understood with the use of multiple eigenvectors (here too, demonstrated via examples in the section following the review of eigenvector centralities).

Eigenvector Centralities

With that general overview of eigenvectors, the social network analyst next considers eigenvector centralities for sociomatrices. Stated simply, the idea behind eigenvector centrality is to give actors more "centrality credit" for being connected to other actors who are, themselves, well-connected. Figure 1 depicts this notion of reflecting both direct and indirect ties. Actors "A" and "B" are of comparable size, which represents similar degree centralities, yet even for actors with comparable degree centralities (or closeness or betweenness centralities), the eigenvector centrality will assign a higher index to actor "B," whose bold ties (for illustration purposes) show connections to actors who are, themselves, highly inter-connected. Actor "A" would have a smaller eigenvector centrality index because the bold ties for this actor are connected to others in the network who are less inter-connected.

To capture such patterns of direct and indirect connections, Bonacich (1972) built on Katz (1953) and proposed that the (first) eigenvector (corresponding to the largest eigenvalue) of an adjacency matrix could serve as such a centrality measure. Bonacich's (1972) idea was that the eigenvectors (of symmetric sociomatrices, and singular value decompositions of asymmetric sociomatrices) would reflect different weighting of ties to partners who, themselves, are highly central versus partners who are less central. Analogous to the example of correlations of survey items, first eigenvectors frequently reflect actors' overall volumes of ties, and it will be demonstrated that second and subsequent eigenvectors reflect other differentiating patterns of the ties.

More precisely, for a $g \times g$ sociomatrix or adjacency matrix on g actors, denoted $\boldsymbol{X} = \{x_{ij}\}$, for actors in rows $i = 1,2, \ldots g$ extending ties to the same set of actors in columns $j = 1,2, \ldots g$, the eigenvector, $\boldsymbol{v}$ (and eigenvalue, λ), are obtained from the familiar equation: $\boldsymbol{Xv} = \lambda\boldsymbol{v}$. The eigenvector score for actor i is $C_{EV}(i)$, a weighted function of the statuses of the other actors to whom actor i is connected: $C_{EV}(i) = x_{1i}v_1 + x_{2i}v_2 + \cdots + x_{gi}v_g$. Katz (1953), suggests norming $\boldsymbol{X}$ to have values of 1.0 for all columns. However, this standardization would negate one of the patterns that is frequently of interest in social networks—the likely different patterns of popularity among the actors in terms of the ties they receive. Hence, these analyses proceed with $\boldsymbol{X}$ with no arbitrary normalization. For simplicity, the adjacency matrix was constructed to be binary and symmetric. However, more complex sociomatrices would only strengthen the case that additional eigenvectors would be informative.

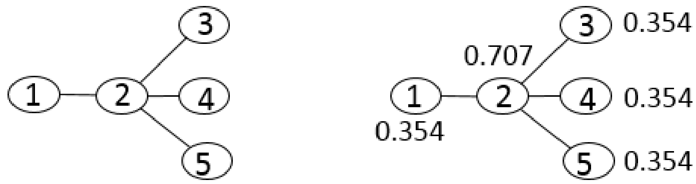
To track the eigenvector centrality on a small example, consider Figure 2. The first network has 5 actors in a star configuration. The eigenvalues of the 5×5 sociomatrix are: 2, 0, 0, 0, −2, and, hence, at most, one eigenvector would be extracted. Given that eigenvalues capture a sense of variability, as soon as they diminish to zero or negative values, those corresponding eigenvectors would not be extracted. The eigenvector numbers are attached to the actor labels at the right, and they reflect the different role of actor 2.

In the second network in Figure 2, there are 7 actors, wherein actors 2 through 5 are connected as previous, but actor 1 now has additional connections. The eigenvalues of this 7×7 matrix are: 2.175, 1.126, 0.000, 0.000, 0.000, 1.126, 2.175, indicating that the representation of this sociomatrix would be helped by two eigenvectors and a focus on only the first eigenvector would be insufficient. To the right of the network, the actors are plotted using their scores on the two eigenvectors. If the second eigenvector had been ignored, the first eigenvector would indicate that actors 1 and 2 are distinct from 3-5 and 6-7, which is accurate and reflective of the volume of ties, or their degrees. However, it is more precise to also include the information in eigenvector 2, the vertical axis, which offers a new perspective on these actors. The 2-dimensional information makes it clear that while actors 1 and 2 are similar in one regard (vector 1), they also play different roles (distinguished along vector 2), and that actors 3-5 are highly similar to each other, as are actors 6 and 7, but different between sets. Together, the two eigenvectors have essentially identified four meaningful blocks of roughly stochastically equivalent actors {1}, {2}, {3, 4, 5}, and {6, 7}. The use of both eigenvectors captures all the nuances of the network.

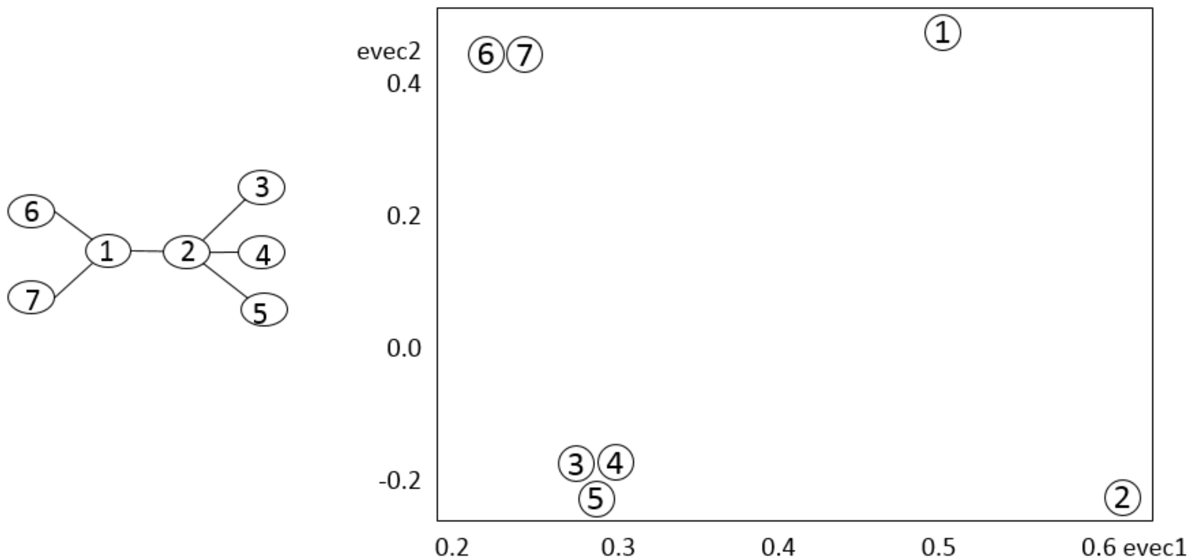
In the sections that follow, additional demonstrations are presented that highlight the potential information contained in second and subsequent eigenvectors. Early research on the use of eigenvectors as centrality scores was focused on making a persuasive case that such a factoring of

Figure 2. Small Examples of Eigenvector Centralities

Example with $g = 5$



Example with $g = 7$

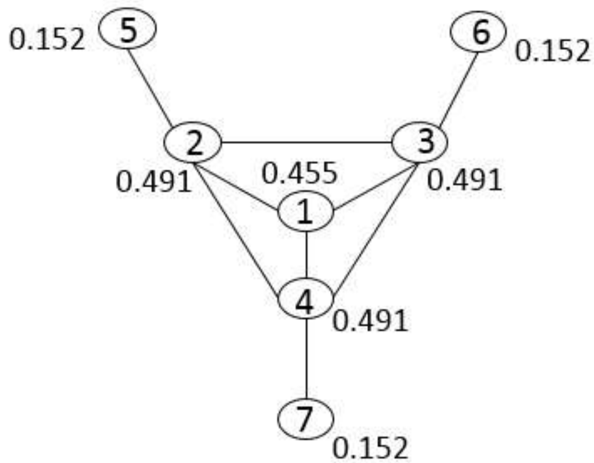


a sociomatrix was useful in a manner analogous to other centralities, such as degree, closeness, betweenness centralities, and, like those established indices, also had a specific objective, with eigenvectors being sensitive to combinations of direct and indirect linkage patterns. That research did not explicitly reject the use of the second or subsequent eigenvectors, but those second and later vectors were also frequently not mentioned (cf., Katz, 1958), though Bonacich (1972) hints at the eigenvectors that follow, and Wright and Evitts (1961), cited therein, explored multiple factors (as did Comrey, 1962), but this extended vector extraction does not seem to have been continued in the literature. In the sections that follow, the utility of multiple eigenvectors for small, hypothetical networks, are examined as well as that for real network data.

One, Two, and Three or More Eigenvector Examples in Small, Hypothetical Networks

If the number of eigenvectors a network analyst should extract depends upon the number of relatively large eigenvalues, it is important to acknowledge that sometimes working with a single eigenvector will be sufficient and appropriate, if that is what the eigenvalues indicate. For example, Figure 3 contains a core-periphery network; that is, there is a subset of actors that are highly

Figure 3. One Eigenvector: Centrality Scores for a Core-Periphery Network

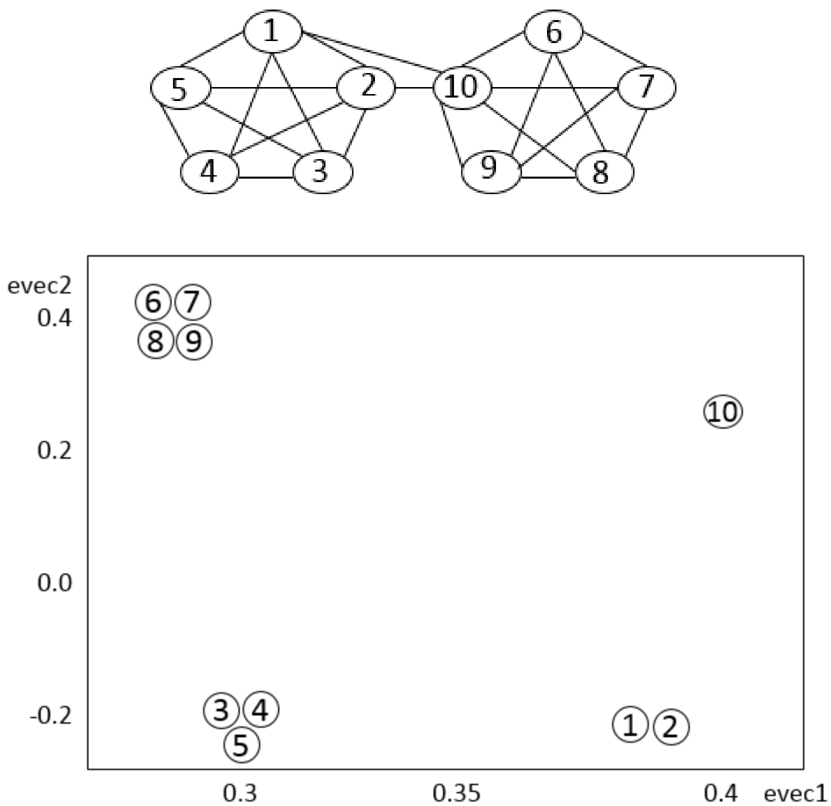


interconnected and a second set of actors connected to the first, but not as completely linked to them, nor to each other (cf., Borgatti and Everett, 1999). The network's eigenvalues are 3.24, 0.62, 0.62, 0, …; the relatively large fall-off from the first to second eigenvalues, along with the equality of the second and third, suggest a single eigenvector is sufficient for capturing the essence of the network. This result is due to the network being very small and very clean in structure. The eigenvector scores are attached to the actors in the figure, and they rather clearly delineate the different roles of the core players versus those along the peripheral edges, in this small example, reflecting essentially volumes of ties.

Figure 4 depicts a slightly more complicated network structure. In it, two cliques are connected by two ties. (Locating cliques was one of the intended uses of eigenvector weights, as described by Bonacich, 1972.) The eigenvalues for this network are: 4.497, 3.678, -0.118, -1, -1, -1, -1, -1, -1, -2.058, suggesting two eigenvectors may be fruitful in representing the actors' positions. Scores on the first eigenvector seem to reflect volume of ties (it is often the case in networks that the eigenvector is at least modestly correlated with degrees), given that actor 10 has six ties, actors 1 and 2 have five, and the other actors have four, and the scores cleanly distinguish the roles of the boundary spanning actors 1, 8, and 10 from the others. In addition, the second eigenvector conveys complementary information and, together, the two eigenvectors locate four sets of actors with similar structures, whereas the use of solely the first eigenvector would have distinguished only two sets of actors. Even for this simple network, had a network analyst relied solely upon the first eigenvector, valuable information would have been lost.

Per helpful suggestions of reviewers, the two sets of eigenvector scores for these 10 actors were correlated with other information. The network is a hypothetical example, but the analysis yielded the following: Firstly, the first eigenvector is significantly correlated with degree (r = 0.955), closeness (r = 0.936), and betweenness (r = 0.936). In contrast, the second eigenvector is not correlated with any of the traditional centrality measures: degree ($r = -0.058$), closeness ($r = -0.047$), or betweenness (r = 0.011). Next, a dummy variable was created to represent the clique in which an actor resides. Specifically, group one was comprised of actors 1-5, and group two was defined as consisting of actors 6-10. The first eigenvector was not significantly correlated with

Figure 4. Two Eigenvectors: Centrality Scores for Two Connected Cliques

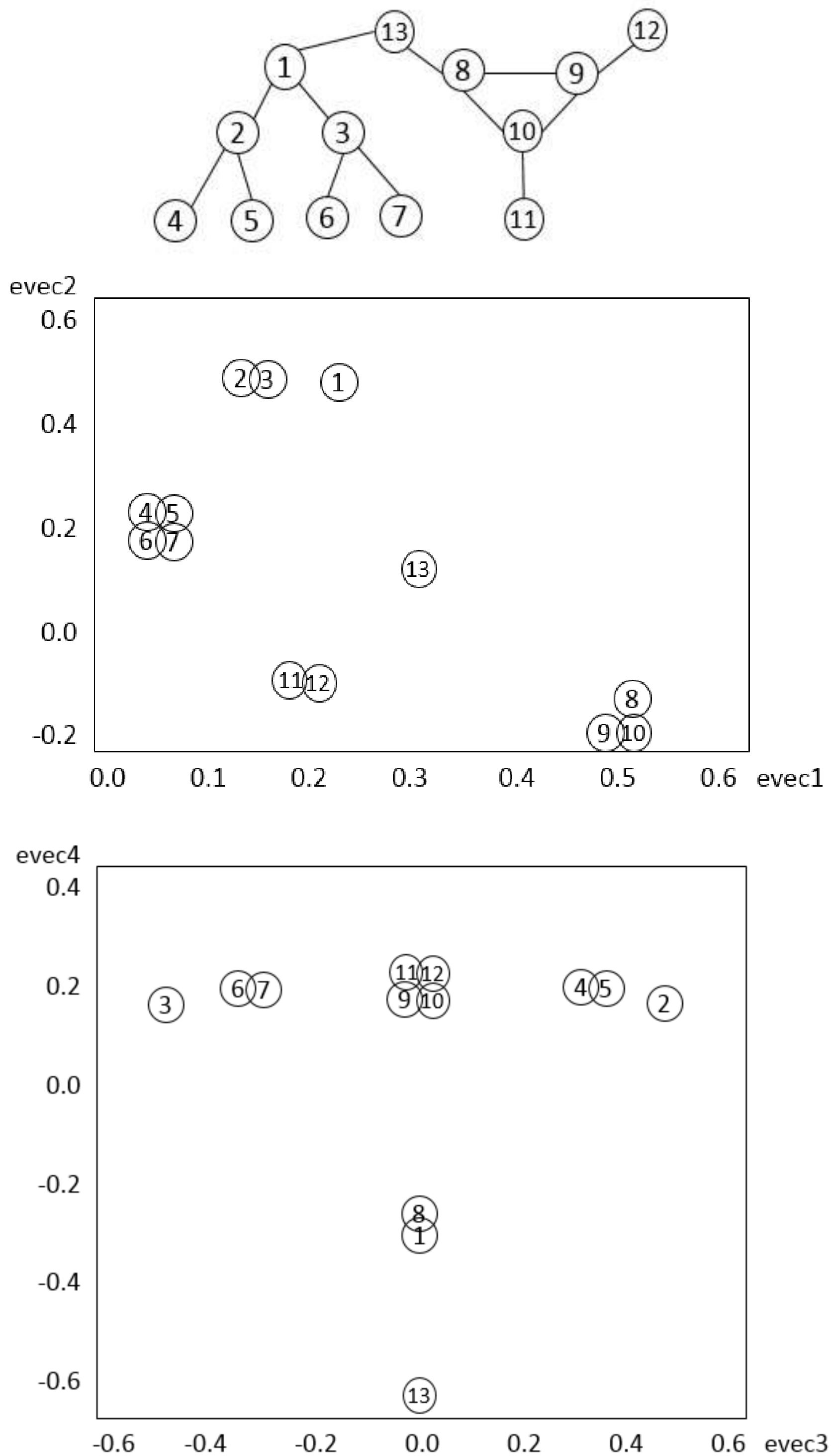


group membership ($r = -0.295$), where group membership seems to be the structural property that the second eigenvector detects (r = 0.993). It is always be helpful in the interpretation of a network analysis to find correlates of all eigenvectors extracted, and it should be the case that, just as happened in this small example, doing so can help make clear what structural patterns a second or third eigenvector may be reflecting.

Figure 5 shows a network that contains two local substructures—a hierarchy and a core-periphery, with one connection linking them. The eigenvalues are 2.47, 2.09, 1.41, 0.86, 0.62, 0, 0, 0, -0.64, -1.41, -1.58, -1.62, -2.20, which suggest that three or possibly even four eigenvectors may be useful. The magnitudes of these eigenvalues show less dramatic delineations between those that are probably associated with substantial eigenvectors versus those that are associated with eigenvectors that essentially convey noise. To proceed, the analysis might begin by examining the four vectors, and if the fourth is meaningful, retain it, and if it does not seem to be interpretable or helpful, retain only the first three. Thus, the information conveyed by the first four eigenvectors is examined. (Recall the intention with this example is to demonstrate an example with more than one or two eigenvectors.)

The first eigenvector in Figure 5 conveys volume information. Specifically, actors 8-10 have the highest eigenvector centralities and three links each, which contrasts to actors 4-7 who have the lowest eigenvector centralities and only one link each. The correlation with degree centralities,

Figure 5. Four Eigenvectors: Centrality Scores for Network Connecting a Hierarchy and Core-Periphery



$r = 0.678$, is not perfect, given that actors 1-3 also have three links. However, their eigenvector scores are a bit lower, but the design of an eigenvector was never intended to be wholly redundant simply with degrees. (A first eigenvector is typically correlated with degree centrality, yet not typically perfectly correlated, both findings are a result of the fact that the first eigenvector is designed as an iteratively weighted function of actor degrees. Thus, it will be typically related to, yet not completely redundant with, degree centralities.) The second eigenvector empirically delineates other pattern information, such as portions of the hierarchy (i.e., actors 1-3 at the top of the hierarchy versus those at the bottom, actors 4-7), the core (actors 8-10) versus the peripheral (actors 11 and 12), and actor 13, the boundary spanner that links the sub-networks.

In the second plot in Figure 5, the third eigenvector further delineates the mapping of the left portion of the hierarchy (actors 2, 4, 5) from the right (actors 3, 6, 7), with the core-peripheral actors sitting at the center of this vector because they do not contribute to this distinction. The fourth eigenvector contrasts actors 1, 8, and 13 (with negative indices) from the rest (whose indices are positive), a difference that is interesting given that actors 1, 8, and 13 are precisely those that play a substantial role in connecting the two local structures. Thus, a network analyst might wish to include the fourth eigenvector as well.

Analogous to the investigation for the network in Figure 4, correlates were sought for the four eigenvectors depicted for the network in Figure 5. The first eigenvector was correlated with degree (r = 0.678), closeness (r = 0.447), and betweenness (r = 0.460). The second eigenvector was not significantly correlated with any of these centrality scores (average r = 0.239). The third eigenvector was not correlated with the traditional centrality scores (all r's = 0.000). The fourth eigenvector was not correlated with degree ($r = -0.333$), but it was significantly correlated with closeness ($r = -0.835$) and betweenness ($r = -0.824$). Those latter two correlations were sufficiently high as to give pause as to whether extracting four eigenvectors was overly much, as the fourth eigenvector may be perhaps redundant with the first. However, the fourth eigenvector was not entirely redundant with the first, as will be addressed and demonstrated shortly.

Next, dummy variables were created to capture other structural properties, specifically which group an actor was in (group 1 was actors 1-7, group 2 was actors 8-13), whether an actor was a spanner (yes for actors 1, 8, and 13), whether the actors existed in a clique (yes for actors 8, 9, 10), whether actors had positions that were moderately between (yes for actors 1, 2, 3), and whether actors were in a subgroup in the hierarchy to the left (subgroup one was actors 3, 6, 7; subgroup two was actors 2, 4, 5). The findings follow:

- The first eigenvector was significantly correlated with the group delineating whether they were a part of the hierarchy or the core-periphery (r = 0.771), and the clique of actors 8-10 (r = 0.893).
- The second eigenvector was correlated with these as well (group $r = -0.854$, and clique $r = -0.711$), and, in addition, reflected the actors who have moderate between positions (actors 1, 2, 3, r = 0.745).
- The third eigenvector reflected subgroup membership (r = 0.795 for the second group of actors 2, 4, 5).
- Regarding the issue of whether the fourth eigenvector was redundant with the first, it was not, as it captured the roles of the spanners (actors 1, 8, 13), which is frequently an important network diagnostic.

These investigations were intended as a reminder that multiple eigenvectors can provide greater information than information contained in only a single eigenvector. Even these small networks demonstrated that while occasionally a single eigenvector may be sufficient, in general, it may be beneficial to extract multiple eigenvectors to enrich the profile of the actors' positions within the networks.

These examples have also been useful in illustrating the frequent observation that the first eigenvector typically reflects volume as correlated with degree centrality. Second eigenvectors, and those that follow, are designed mathematically to be orthogonal to, or uncorrelated with, the first eigenvector, and, hence, are less likely to be correlated with standard network centrality indices, such as degree. The information provided by the second eigenvector in Figure 4, and the second, third, and fourth eigenvector in Figure 5 distinguished roles of actors in networks in a manner more detailed than a reflection of volume, as important as volume is. In the section that follows, it will be shown that in real (typically noisy) network data, a single eigenvector will seem to be insufficient for fully capturing the information in the network.

Eigenvectors on a Real Social Network

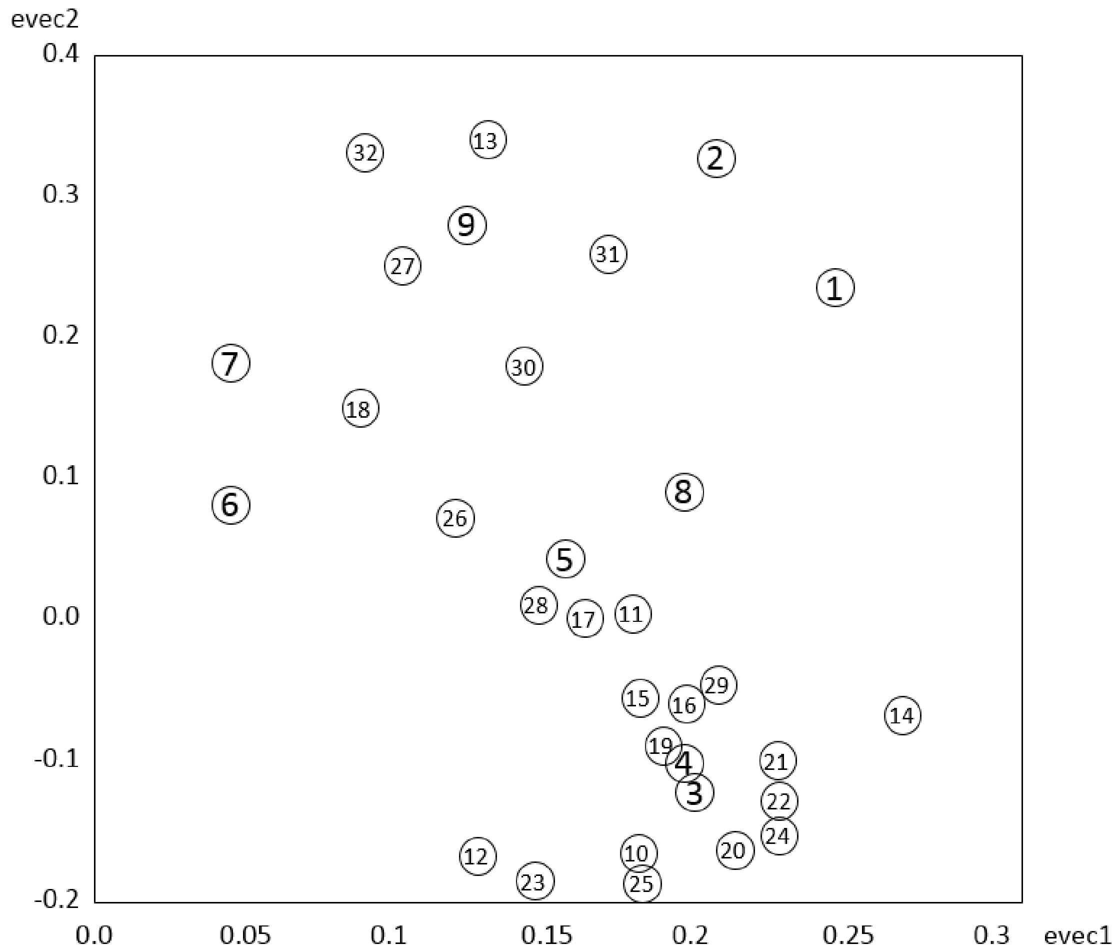
In this section, the question is posed as to whether the multiple eigenvectors issue matters on a real social network. The analysis examines the 32 actors in Freeman's EIES (electronic information exchange system) network (Freeman and Freeman, 1979, as measured at time 1), and it shall show be shown that a single eigenvector would not provide a complete analysis of the patterns of the social connections.

Freeman EIES Network

The 32×32 EIES network (Freeman and Freeman, 1979) was symmetrized by averaging the sociomatrix values; $\boldsymbol{X}_{sym} = \frac{1}{2}(\boldsymbol{X} + \boldsymbol{X}')$. This matrix yielded ordered eigenvalues that begin: 46.89, 13.99, 8.54, 5.52, 3.93, 2.32, 1.87, 1.14, 0.27, 0.08, -0.16, and suggest that two eigenvectors would be sufficient in capturing most of the network patterns. It may be the case that a network scholar might believe that three (or more) eigenvectors would be necessary to capture the essence of the patterns in the matrix. If there is uncertainty, the third vector can be examined to see if its inclusion is necessary for the data description (or whether two eigenvectors may be sufficient), considering the slight loss in parsimony if one were to proceed with three rather than two eigenvectors. Each eigenvector could be correlated with any additional measures on the actors—their positions or their attributes, to search for significant correlates and explanations of subsequent vectors, which would strengthen the case for keeping them. Regardless, recall once again that the main point is that in many applications, one vector might not be sufficient, so two or three, or more, may be beneficial.

Figure 6 contains the plot of the 32 actors along the two eigenvectors. Without knowing any content to describe the network, the fact that there is scatter in this plot indicates that there is more

Figure 6. Freeman Time 1, Two Eigenvectors



variability among these actors than was captured solely along the first eigenvector. The Freeman EIES network is familiar to social network scholars, but its characterization by eigenvector centrality has been insufficient. By definition, a second eigenvector brings new information to the network analysis beyond the first.

Freeman's EIES network was selected as an illustration in part because data on actor attributes are available, and these might help interpret the eigenvector centralities. The actor attributes include a researcher's citation count and a dummy variable for the researcher's discipline (sociology, anthropology, statistics, or psychology). Thus the correlations among these eigenvectors, the actor attributes, and the other traditional centralities of degree, closeness, and betweenness were examined.

The correlations in Table 1 show the typical finding that the first eigenvector is rather highly correlated with three standard measures of actor centrality: $r = 0.95$ for degree, $r = -0.59$ for closeness, and $r = 0.62$ for betweenness. (Also not unusual in real networks, these three indices were somewhat correlated amongst themselves: $r_{degree,close} = -0.49$, $r_{degree,between} = 0.69$,

Table 1. Freeman Time 1 Network Correlates

	Eigenvector Centralities	
	1	2
Degree	0.95***	.
Closeness	-0.59*	.
Betweenness	0.62***	.
Actor Attributes:		
Citations	.	-0.48**
Sociology	.	-0.60*
Anthropology	.	0.64***
Statistics	.	.
Psychology	.	.

*p<.01, ***p<.0001, "." = n.s.

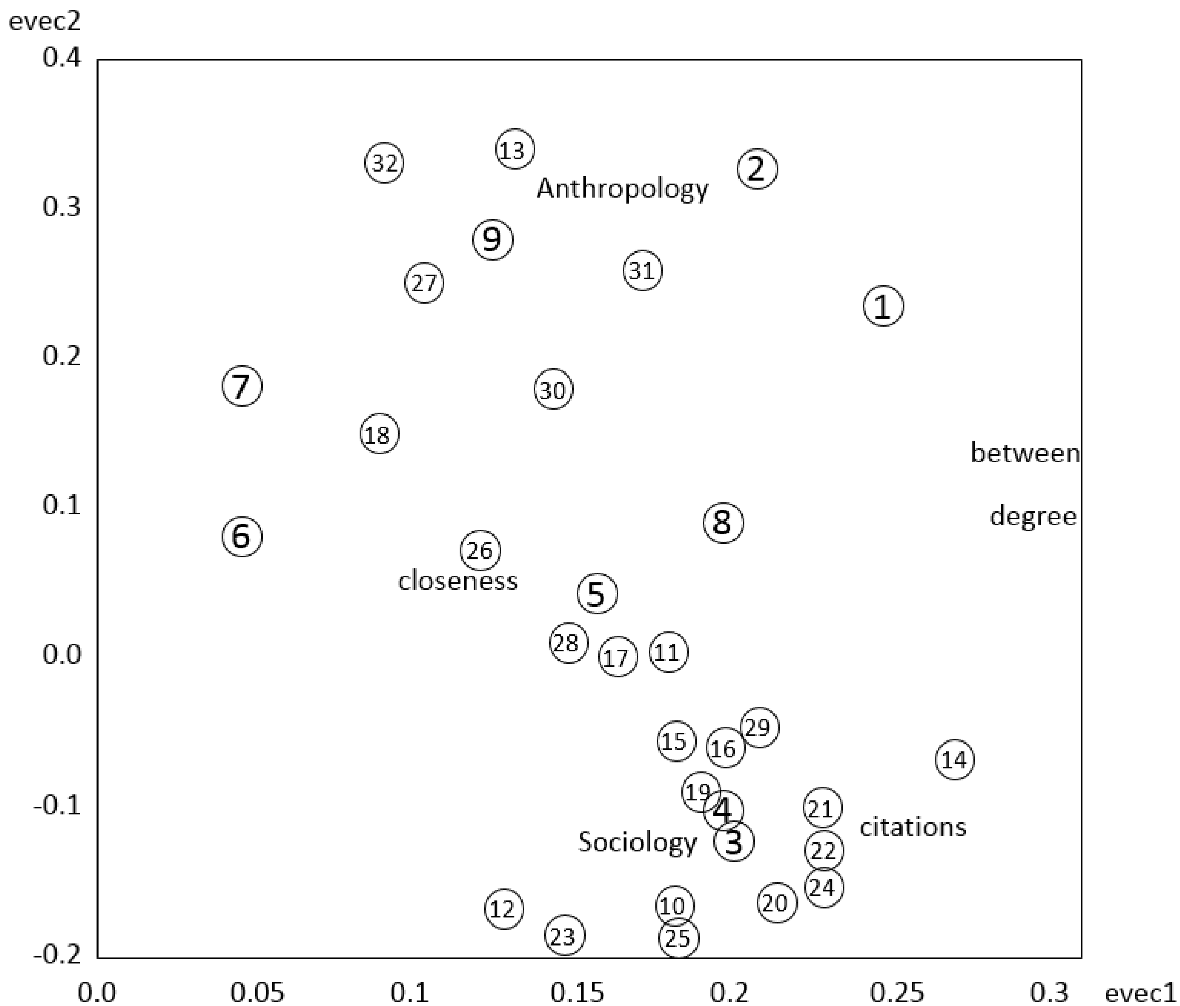
and $r_{close,between} = -0.29$.) In contrast, the second eigenvector was not at all redundant (not significantly correlated) with the three other measures of actor centrality.

Next, note that the first eigenvector does not correlate with any of the actor attributes, nor did degree centrality, closeness, or betweenness centrality correlate with the actor attributes. However, the second eigenvector picks up an inverse relationship to citations and belonging to sociology, and a positive association with anthropology.

Figure 7 depicts the relationship between the eigenvectors and the actor attributes. Using regression to determine the location of the attributes (cf., Davidson 1983), the group of actors in the "south" of the plot tend to be the sociologists, who are also more heavily cited, whereas the anthropologists collect at the "north" of the plot. The correlations with the other disciplines, statistics and psychology, were not significant. If they were to be represented, their locations would be placed at the origin of the plot. Next, the "east" of the plot is marked by the degree and betweenness centralities, consistent with their correlations with the first eigenvector (i.e., the east-west axis). Closeness is also correlated with the first eigenvector, but the correlation is negative, hence it is more towards the "west" of the plot.

Finally, as a check on the analyses, a singular value decomposition was also derived on the original Freeman Time 1 network which was asymmetric; per $\boldsymbol{X} = \boldsymbol{LAC}$, where $\boldsymbol{L}$ contains the eigenvectors of $\boldsymbol{XX'}$, the matrix describing the similarities among the actors' outgoing tendencies having aggregated over their partner behaviors, $\boldsymbol{C}$ contains the eigenvectors of $\boldsymbol{X'X}$, the matrix describing the partners' receiving tendencies, having aggregated over the actor initiatives, and $\boldsymbol{A}$ contains the singular values which are the square roots of the eigenvalues of the $\boldsymbol{XX'}$ and $\boldsymbol{X'X}$ matrices. The first eigenvector of $\boldsymbol{X}_{sym}$ (plotted previously) was highly correlated with the first vector of $\boldsymbol{X}$ in $\boldsymbol{L}$, $r = 0.92$, and the second eigenvector of $\boldsymbol{X}_{sym}$ was highly correlated with the second vector in $\boldsymbol{L}$, $r = 0.97$. Similarly, the first eigenvector of $\boldsymbol{X}_{sym}$ was highly correlated with the first vector in $\boldsymbol{C}$, $r = 0.95$, and the second eigenvector of $\boldsymbol{X}_{sym}$ was highly correlated with the second vector in $\boldsymbol{C}$, $r = 0.97$. Thus, little information seems to have been lost by treating the network as essentially symmetric with mutual ties.

Figure 7. Freeman Time 1, Two Eigenvectors, with Actor Attributes



Discussion

When using eigenvector-based methods, such as principal components or factor analysis, social and physical scientists often extract more than one vector or factor. To characterize one's data otherwise, is to leave much of it unexplained. In this paper, illustrations have been offered to help support the recommendation that an extension beyond a single eigenvector should also apply to the analysis of social networks.

In the social networks literature, the traditional emphasis is to focus on extracting a single eigenvector to represent a centrality index. The current research considered whether the centrality information derived from a first eigenvector is sufficient for capturing structure contained in social networks. It was shown, in hypothetical and real data, that subsequent eigenvectors could provide supplemental information.

When social scientists extract principal components or factors, no one would think to necessarily limit themselves to a single factor (i.e., one is rarely sufficient). However, in typical application, social networks scholars focus solely on the first eigenvector. It was shown that additional eigenvectors may be informative in the world of social networks, and, therefore, they should also be extracted and used for a richer understanding of the structure in the network. The first eigenvector will typically be correlated with traditional measures of centrality, particularly degree. Extracting a second, third, or more eigenvectors will necessitate further investigation as to the nature of the structural patterns that the new eigenvectors reflect. Even in the simple networks depicted in Figures 4 and 5, but also in the real EIES network depicted in Figures 6 and 7, several classes of network structures and actor attributes were shown to have mapped onto the eigenvector scores. This second step of analyses, used to help interpret the eigenvector scores, required calculating correlation coefficients.

The first eigenvector, in any statistical application including the analysis of social ties, meets the objective function of explaining the maximum amount of variance in the dataset. The second eigenvector is derived to explain the maximum amount of remaining variance, subject to the constraint that the resulting vector be orthogonal to, or uncorrelated with, the first eigenvector. Thus, in social network analysis, while the first eigenvector centrality index is likely to be correlated with the degree, closeness, and betweenness centralities, a multi-dimensional eigenvector centrality, including the second eigenvector, and, if necessary, those that follow, will be uncorrelated with the previous eigenvector(s) and therefore uncorrelated with the traditional degree, closeness, and betweennness centralities as well. This lack of redundancy indicates the supplemental information that the second and subsequent eigenvectors will bring to the network modeler.

As when scholars use eigenvalues and eigenvectors in other arenas (e.g., principal components or factor analysis), network scholars will have to balance the tradeoff of a more thorough understanding of the data (in extracting more eigenvectors) and parsimony (in extracting fewer). In some datasets, it may be the case that only a single eigenvector would be necessary to capture the essence of the network (i.e., if the size of the first eigenvalue greatly dominates the others). However, if two or more eigenvalues are large, relative to the others, it may prove beneficial to examine whether the additional eigenvectors provide enlightening complementary information. If the eigenvalues are only subtly different, it may be that the network scholar concludes that extracting an additional eigenvector is not “worth it” considering the trade-off between the additional value of more information explained versus the added complexity and reduced parsimony.

Note that this research would also have implications for other centrality indices that are based on eigenvectors, such as Bonacich’s power index (1987; 2007; Bonacich and Lloyd, 2001), and Google’s Page Rank index (Brin and Page, 1998; Friedkin and Johnsen, 1990; Friedkin and Johnsen, 2014). The eigenvector-based models have been expanded (e.g., for asymmetries, Bonacich and Lloyd, 2001; and for non-binary and negative values, Bonacich, 2007), and further developed, finessing parameters of the eigenvector values to weight indirect ties to a greater or lesser extent (Bonacich, 1987), and each of these could be generalized as well.

As a practical matter, given the focus of social network analysis on solely the first eigenvector, network scholars seeking to examine second, third, and subsequent eigenvectors will have to circumvent network analysis packages. Network scholars seeking to extract multiple eigenvectors can use software coding such as that provided in Appendix B.

This research described and illustrated the usefulness of a second, third, and possibly additional eigenvectors, beyond the typical extraction and use of only the first eigenvector to capture centrality and social network structural properties. If more than one eigenvalue is relatively large, then the set of multiple eigenvectors contain more information than just the first, which tends to reflect overall volume like degree centralities. In such cases, if the additional eigenvectors are not used, information would be lost in the representation and understanding of the network patterns.

Social network data can be more challenging and effortful to gather than survey data which can seem rather more straightforward. Accordingly, as much information should be extracted from the network data as possible, and additional eigenvectors can enable this goal.

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Appendix A

This appendix provides a refresher that demonstrates the traditional use of eigenvectors on correlation matrices. Contrast two examples. First, Figure A1 presents a correlation matrix of six variables that are all somewhat correlated. This pattern is not unusual when the variables all represent slightly different wordings of a single, underlying concept being measured on a survey. For example, say x1 is a survey item that asks, “How satisfied are you with today’s flight?” and x2 asks, “How likely is it you would recommend our airline to your friends?” through to x6 which asks, “How likely is it you will return to our airline the next time you need to fly?” These six questions have much in common and are likely correlated; as one element of customer satisfaction rises, others will likely follow suit. The eigenvalues of the matrix in Figure A1 (3.675, 0.578, 0.500, 0.500, 0.500, 0.247) show a sharp decline in magnitude after the first, suggesting that a single eigenvector will capture the majority of the variance in the data. The entries in the eigenvector indicate that the optimal means of combining these variables is essentially an average (i.e., weighting x1 by 0.449 through x6 by 0.385).

By comparison, Figure A2 shows a correlation matrix among six variables in which two latent concepts seem to have given rise to the data, one concept driving x1–x3, another for x4–x6. For example, perhaps x1–x3 are “satisfaction” questions as suggested previously, whereas in this survey, perhaps the question x4 asks, “Do you believe the cost of your air travel was fair?” through to x6 that might ask, “Do you think the price of your airline ticket was good value?” For these new six questions, x1–x3 will be highly correlated, each tapping the construct of satisfaction, and x4–x6 will be highly correlated, each reflecting a price assessment. Naturally, the two sets are modestly correlated. The eigenvalues for this correlation matrix (3.052, 1.562, 0.408, 0.401, 0.292, 0.286) show a dramatic decline after the second, indicating the extraction of two eigenvectors would be more fruitful than that of a single vector. The first eigenvector again suggests that much of the structure of the matrix would be captured simply by an average of the six variables. The second eigenvector delineates the two groups, with variables x1–x3 having negative coefficients, and x4–x6, positive. (In principal components and factor analyses, of course, these initial matrices are usually rotated to further clarify the structures, however the goal of simple structure, objectively maximized by a a preponderance of zeros in the rotate matrix to represent constructs, usually of variables, here of actors, seems less applicable, but certainly, in some uses, social networks scholars may find a reason to do so; Kim and Mueller, 1978). If this analysis had proceeded with only the first eigenvector, obviously the information contained in the second vector would have been lost.

Table A1: Correlation Matrix with One Underlying Construct

	x1	x2	x3	x4	x5	x6	eigenvector
x1	1.00						0.449
x2	0.70	1.00					0.414
x3	0.65	0.50	1.00				0.406
x4	0.60	0.50	0.50	1.00			0.399
x5	0.55	0.50	0.50	0.50	1.00		0.392
x6	0.50	0.50	0.50	0.50	0.50	1.00	0.385

Table A2: Correlation Matrix with Two Underlying Constructs

	x1	x2	x3	x4	x5	x6	vector 1	vector 2
x1	1.00						0.416	-0.421
x2	0.70	1.00					0.409	-0.406
x3	0.65	0.60	1.00				0.401	-0.387
x4	0.20	0.20	0.20	1.00			0.392	0.473
x5	0.25	0.25	0.25	0.70	1.00		0.407	0.412
x6	0.30	0.30	0.30	0.65	0.60	1.00	0.423	0.339

In this paper, it was argued that this issue exists in analogous form for eigenvector centralities of social network data. That is, sometimes one eigenvector can be sufficient, but often, additional eigenvectors can provide useful complementary information.

Appendix B: SAS Code to Obtain Multiple Eigenvectors

```
proc iml;
x={ 0 1 0 0 0,
       1 0 1 1 1,
       0 1 0 0 0,
       0 1 0 0 0,
       0 1 0 0 0 }; *This matrix is the first in Figure 2.;
val=eigval(x); vect=eigvect(x); print val vect;
quit; run;
```

This code will derive the eigenvectors of the sociomatrix $\boldsymbol{X}$. Alternatively, symmetric sociomatrices could be submitted to principal components analyses in packages such as SPSS or SAS. A principal component is merely an eigenvector with each element multiplied by the square root of its eigenvalue. That is, a first component begins with the first eigenvector and multiplies each element by the square root of the first eigenvalue (and a second component is the second eigenvector multiplied by the square root of the second eigenvalue, etc.). This multiplication essentially stretches the results to resemble ovals, with greater variance on the first axis than on the second (to reflect that $\lambda_1 > \lambda_2$). Strictly speaking, the resulting components from SPSS or SAS should be scaled back to return to their original eigenvectors. To obtain the original eigenvectors, the elements in the components loadings matrix would be divided by the square roots of their respective eigenvalues. However, given that one is a function of the other, they are perfectly correlated, so reporting two or three components rather than two or three eigenvectors would not be misleading, because the ordering of the actors along either the vector or the component would be the same.