

**Comment & Response on [Choosing the 'β' Parameter when Using the Bonacich Power Measure](#)**

**Comment by  
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**Response by  
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## Comment by Phillip Bonacich

Understandably, I feel a personal responsibility toward maintaining a clear and accurate public understanding of the Bonacich Power Measure. I read the recent paper “Choosing the ‘ $\beta$ ’ Parameter When Using the Bonacich Power Measure,” by Simon Rodan (2011), with great interest, since it is a commentary and critique of a paper of mine (1987). To briefly recap the 1987 paper, a family of centrality measures was proposed.

$$c_i(\alpha, \beta) = \sum_j (\alpha + \beta c_j) R_{ij} \quad (1.1)$$

$R$  is the adjacency matrix.  $\alpha$  is an unimportant scaling constant,  $\beta$  reflects the effects of the centrality of its neighbors on a node’s centrality, and  $u$  is  $R$ ’s largest eigenvalue. When  $|\beta| < 1/u = \kappa$ , the matrix solution to equation (1.1) is:

$$c(\alpha, \beta) = \alpha(I - \beta R)^{-1} R \mathbf{1} \quad (1.2)$$

The point of the 1987 paper was to show that two types of network status were not radically different and could be unified conceptually into one family of measures generated by the value of just one parameter  $\beta$  – reflecting how ego’s status is affected by the status of his neighbors. For access to information, popularity, or social status this effect would be positive. For exchange networks of the sort studied by Karen Cook (1983) and others using a power-dependence orientation, having weak neighbors with no alternative exchange partners is a source of power.

Rodan examines some of the reasons why network values are sensitive or insensitive to the choice of  $\beta$ . Sensitivity is assessed by the correlation  $\Psi$  between  $c(\alpha, -\kappa)$  and  $c(\alpha, \kappa)$ . For example, centrality scores for high density networks tend to be insensitive to the value of  $\beta$ . The paper suggests that pendants (vertices with just one connection to others) drive differences in centrality scores. It also suggests that often  $c(\alpha, \beta)$  need not be computed at all because it is usually a linear combination of its two extreme values

There are a few small errors. On page 3 Rodan writes that  $\Psi$  is “exactly” proportional to  $1/N$  for chains of even length. When I look at even values of  $N$  between 4 and 12 I don’t find exact proportionality. I get the following results: ( $N=4, \Psi=1$ ); ( $N=6, \Psi=.590$ ); ( $N=8, \Psi=.424$ ); ( $N=10, \Psi=.335$ ); ( $N=12, \Psi=.280$ ). Close, but not exact. In Table 1  $\beta$  varies between  $-1$  and  $+1$  rather than from  $-\kappa$  to  $+\kappa$ . The equation for Bonacich centrality (equation 1 in Rodan’s paper) is incorrect; the minus on the right side of the equation should be a plus. The reader may also be confused by the apparent repeated calculation of  $c(\alpha, \beta)$  when  $\beta=\kappa$ ; Equations (1.1) and (1.2) actually have no solution when  $\beta=\kappa$ . Only in a footnote is this clarified.

It is questionable that the correlation coefficient  $\Psi$  is a good measure of sensitivity. Consider the three person chain  $i \leftrightarrow j \leftrightarrow k$  that Rodan introduces at the beginning of the paper. As Rodan points out, changing  $\beta$  from  $+\sqrt{2}/2$  to  $-\sqrt{2}/2$  has a profound effect on  $c(\alpha, \beta)$ . For the positive  $\beta$  the values are (.5, .71, .5) but for the negative  $\beta$  the  $i$  and  $k$  scores are much lower than the  $j$  score, (-.5, .71, -.5),

reflecting  $i$ 's and  $k$ 's vulnerable and dependent positions. But,  $\Psi=1$ . What might work better is the slope of a regression line with just one free parameter and no intercept. In the 3 person chain, the one parameter slope is zero.

My other points of disagreement are much more fundamental. Rodan divides networks into two types: those whose centrality scores are insensitive to  $\beta$ , and those that are sensitive. The classification is based exclusively on the typology of the network. The former type has short distances between vertices, is dense, and has few pendants. In these networks "the choice of  $\beta$  is unlikely to matter." The 1987 paper also classified networks into two types but the classification was based on the substantive characteristics of the relationship the network described rather than on the formal properties of the graph. Networks in which power accrues to a node connected to other well-connected nodes should be modeled with  $\beta>0$ . Networks in which individuals can exploit others who have no or few alternatives (the "power-dependence" argument) should be modeled with  $\beta<0$ .  $\beta$  could be treated as a parameter to be estimated from the data by correlating centrality scores with independent measures of power, but getting the "wrong" sign has to mean that the study has been based on fundamentally incorrect assumptions. Using a negative value of  $\beta$  in a study of high school popularity would be wrong no matter how insensitive  $c(\alpha,\beta)$  was to values chosen for  $\beta$ .

The scores at the extremes are mislabeled as the outcomes "competitive" ( $-\kappa$ ) and "cooperative" ( $+\kappa$ ) strategies; for actors dominated by competitive motivations the minimum value of  $\beta$  is called for, and if actors are cooperative with one another the maximum value is appropriate. However,  $\beta$  was intended to capture not strategies or motivations but rather the effectiveness of transmission of a resource between connected neighbors. The two extremes are meant to represent situations in which status is borrowed from one's neighbors (popularity or information) or is inhibited by the status of one's neighbors (power-dependence and exploitation). College athletic teams achieve high ranking in polls by playing highly ranked opponents. Strength of schedule is a component in the football Bowl Champion Series ranking. The relationship is competitive but a positive value of  $\beta$  would be appropriate.

Rodan stresses the importance of pendants in determining centrality scores regardless of the value of  $\beta$ , but the reader could be misled into thinking that pendants are the sole factor driving centrality score differences. Consider, for example, a bipartite graph (one in which all contacts are between rather than within two classes), with  $N_1$  and  $N_2$  vertices in the two classes,  $N_1>N_2>1$ , all  $N_1\times N_2$  ties existing between classes, and a negative  $\beta$  (the two classes represent buyers and sellers of a commodity). The differences in scores between the two categories will be large and dramatic even though there are no pendants.

Finally, I agree in part with Rodan's conclusion that  $c(\alpha,\beta)$  should usually be calculated at its extreme values. When the transfer of status is positive but the researcher has no interest in the particular value of  $\beta$  it is best to calculate an eigenvector. In power-dependence situations it's best to calculate  $c(\alpha,-\kappa)$ . I don't agree with Rodan that the meaning of  $\beta$  is "far from clear." The parameter  $\beta$  does have a meaning. It gauges the degree to which the status of actors is affected by the status of their neighbors.

How then should one choose  $\beta$ ? The choice of a positive or negative value will ordinarily be obvious. Is one studying a resource that diffuses through a network (information, status, popularity), or is one studying a power-dependence relation in which actors can exert power over neighbors with few alternatives? After this decision, the best value of  $\beta$  may be estimated from correlations with other measures of power or status, especially if one wishes to compare networks on this value. Alternatively, one may choose an extreme value of  $\beta$  depending on whether one is interested in the flow of status or power-dependence. There is a third alternative for positive values of  $\beta$ . If  $\beta$  is interpreted as the probability that a resource will be transferred between neighbors, then  $1/(1-\beta)$  is the mean length of indirect and direct transmissions. For example,  $\beta=1/3$  implicitly assumes that an individual's status is affected by those three links away.

### **References**

Bonacich, P. (1987). "Power and Centrality: A Family of Measures." *The American Journal of Sociology* 92, 5:1170-1182.

Cook, K. S., R. Emerson, M. R. Gilmore, and T. Yamagishi. (1983). "The Distribution of Power in Exchange Networks: Theory and Experimental Results." *American Journal of Sociology*. 89, 2:275-305.

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## Simon Rodan responds

Let me begin by thanking Professor Bonacich for taking the time to write a detailed and constructive commentary.

At the outset, I'd like to clarify that the aim of the paper was not to refute the Bonacich measure's theoretical or mathematical underpinning; indeed, the unification of the two effects, power (adversarial ties) and status (cooperative ties) in a single measure is extremely elegant. Rather, my goal was to provide researchers with some practical guidance when looking at networks with adversarial or cooperative ties, or a mixture of the two. Let me turn in order to the points Bonacich makes.

First, additional investigation shows that the 'exactness' of the relationship between  $\psi$  and chain length turns out to be very sensitive to how close  $\beta$  is to the extreme values:  $\beta = 0.99 * |\kappa|$  gives rather different results from  $\beta = 0.99999 * |\kappa|$  as illustrated in Figures 1a and 1b. However, even at the highest precision possible with x86 architecture, the exponents in  $\psi \propto N^x$  do not converge to integer values as I had assumed; for even length chains,  $x \sim -1.058$  and for odd lengths,  $x \sim -2.075$ . Whether this has any practical implication for real world networks is doubtful.

Second, the reference to the range of  $\beta$  values as  $(-1,+1)$  was a mistake stemming from the coding in my numerical methods; I use  $-1$  and  $+1$  as a convenient 'shorthand' for the two ends of the parameter range enabling me to write the function call as:

$$v = \text{bonacich\_power}(A,-1) \text{ or } v = \text{bonacich\_power}(A,1)$$

without having to first calculate the eigenvalues. The eigenvalues are calculated in the function subroutine, where the second argument value is multiplied by  $+\kappa * 0.999999$ .

Third, Bonacich is also correct to point out that at  $-\kappa$  and  $+\kappa$  there is no solution; Relegating this observation to a footnote as I did may have been confusing.

Fourth, with regards the measure of similarity between  $P_\kappa$  and  $P_{-\kappa}$  my choice to use correlation stemmed from my approach to regression estimation: multicollinearity is a problem when two independent variables are highly correlated and I generally look for this in the pair-wise correlations before running a regression.

The suggestion of an alternative measure based on the constrained regression of  $P_\kappa$  on  $P_{-\kappa}$  (which I will refer to as  $\gamma$ ) is interesting so I ran an experiment to investigate the efficacy of  $\psi$  and  $\gamma$  as predictors of the ability to distinguish positive from negative flows in a network. This experiment, comprising 1,000 trials, involved simulating a dependant variable,  $Y$ , constructed as a linear combination of  $P_\kappa$ ,  $P_{-\kappa}$  and an error, and testing the accuracy with which regression estimation could recover the proportions of  $P_\kappa$  and  $P_{-\kappa}$  from  $Y$ .

In each trial, I constructed a network and calculated  $P_\kappa$  and  $P_{-\kappa}$ . I created a new vector,  $P_m$ , formed as a linear combination of  $P_\kappa$  and  $P_{-\kappa}$ :

$$P_m = \alpha * P_\kappa + (1-\alpha) * P_{-\kappa}$$

Next I generated a simulated dependant variable,  $Y$ , that was correlated with  $P_m$ . The correlation of  $Y$  with  $P_m$  was chosen randomly from the range  $[0,1]$ . I then regressed  $Y$  against  $P_\kappa$  and  $P_{-\kappa}$  and looked at the values of the coefficients in the regression:

$$Y = b_0 + b_1 * P_\kappa + b_2 * P_{-\kappa} + \varepsilon$$

If the regression successfully recovers the contributions of the two components,  $P_{\kappa}$  and  $P_{-\kappa}$ ,  $b_1$  should be equal to  $\alpha$  and  $b_2$  to  $(1-\alpha)$ . I constructed a variable,  $\delta$ , to measure the departure from this ideal:

$$\delta = \sqrt{(b_1 - \alpha)^2 + (b_2 + \alpha - 1)^2}$$

Finally I correlated the two measures,  $\psi$  and  $\gamma$  with  $\delta$ ; the results, along with some additional pair-wise correlations, are shown in Table 1. The distributions of  $\psi$  and  $\gamma$  are shown in Figures 2a and 2b<sup>1</sup>.

**Table 1**

	Mean	Std dev	1	2	3	4	5
1 $\delta$	0.46	0.41					
2 $\psi$	0.20	0.23	0.33 ***				
3 $\gamma$	0.05	0.18	0.09 **	0.57 ***			
4 $c$	0.51	0.29	-0.39 ***	0.04	0.01		
5 $\alpha$	0.50	0.28	-0.12 ***	-0.01	0.02	0.01	
6 $N$	14.82	8.00	-0.21 ***	-0.48 ***	-0.22 ***	0.02	0.02

Both  $\psi$  and  $\gamma$  are related to  $\delta$  in the expected direction. Larger values of each are associated with larger errors of estimation. Table 1 also shows that  $\delta$  is also negatively related to network size,  $N$ . This suggests that smaller networks lead to larger estimation errors; given that  $\psi$  and  $\gamma$  are negatively related to networks size (see also Figure 1), the apparent relationship between  $\delta$  and  $\psi$  or  $\gamma$  might simply be a function of  $N$ . To test this I regressed  $\delta$  against  $\psi$  controlling for network size and did the same for  $\gamma$ . Table 2 shows the results.

**Table 2**

	Model 1	Model 2	Model 3	Model 4
Intercept	0.343 *** (0.017)	0.402 *** (0.038)	0.451 *** (0.014)	0.605 *** (0.03)
$\psi$	0.581 *** (0.056)	0.528 *** (0.064)		
$\gamma$			0.211 ** (0.076)	0.114 (0.077)
$N$		-0.003 † (0.002)		-0.010 *** (0.002)
P [F > Fcrit]	0.000	0.000	0.006	0.000
Adj R Sq	0.109	0.111	0.008	0.044

$N = 857$ , †  $p < 0.1$ , \*  $p < 0.05$ , \*\*  $p < 0.01$ , \*\*\*  $p < 0.001$

Model 1 shows  $\delta$  regressed against  $\psi$  alone, Model 2 adds network size. The significance of the coefficient for  $\psi$  remains strong. Model 3 shows  $\gamma$  alone and Model 4 adds the control for size; in Model 4  $\gamma$  is no longer significant. The relationship between  $\gamma$  and  $\delta$  would appear to be a function of a common driver,  $N$ . My conclusion from this exercise is that  $\psi$  is probably a better indicator of whether  $P_{\kappa}$  and  $P_{-\kappa}$  are sufficiently distinct to allow them to be used independently.

<sup>1</sup> The values of  $\gamma$  were highly clustered towards very small values ( $\sim 10e-5$ ). The graph of  $\ln(\gamma)$  is used to see more of the smaller end of the distribution and is equivalent to using exponentially larger bin sizes.

Now to the more substantive points: Bonacich points out that the value of  $\beta$  should be determined theoretically ex-ante rather than empirically based on the network structure. I don't disagree: I was, however, attempting to make two slightly different points. First given the topology of a network, irrespective of the nature of the ties, it may be empirically impossible to distinguish between the two types, simply because the two extremes,  $\beta \rightarrow -\kappa$  and  $\beta \rightarrow \kappa$ , give essentially identical results. It is therefore useful to be aware that, in some cases, not only is the choice of  $\beta$  moot but that the measure is not open to its normal interpretation. Suppose I believe that association with powerful alters is harmful and I choose a negative  $\beta$ . If the results are significant I would interpret this as confirmation that it is bad to be connected to powerful others. Yet this is exactly the opposite conclusion from that which I would have drawn had I initially made the assumption that association with powerful others was beneficial. That possibility makes it important to know how closely related  $P_\kappa$  and  $P_{-\kappa}$  are when starting an empirical analysis.

Bonacich makes an important point when he notes that  $\beta$  is intended to capture the flow of influence in a network rather than agent motivation; in retrospect, I was clearly guilty of conflating the two and his example of a competitive league is illuminating. I had made the unwarranted assumption that resource flows are always a function of agent motivations and behavior; while that may be the case in many settings such as organizations and firms, Bonacich is right to draw attention to this since, as his example illustrates, the assumption does not apply universally.

Regarding pendants, I did not mean to imply that they are the only cause of low values of  $\psi$ . As Bonacich's example shows, even in networks with no pendants,  $\psi$  can be low. The point I was trying to make was only that pendants are one of several structural features that increase the separation between  $P_\kappa$  and  $P_{-\kappa}$  and hence the sensitivity of the power indices to the value of  $\beta$ . My purpose in discussing pendants, density and link nodes was twofold: first to help illustrate the way in which structure influences 'flows' and second to suggest some things to look for when beginning the analysis of a network.

Bonacich's example also suggest another feature of structure I had not considered that might influence  $\psi$ ; the variation in node degree. In a sample of 1,000 randomly generated networks, the correlation between  $\psi$  and the range of node degree ( $\max(\text{degree}) - \min(\text{degree})$ ) was about 0.39. Table 3, which reprises the table of the same number in the paper, shows how the range in degree compares to the other three predictors of  $\psi$  mentioned in the paper. As expected, higher variation in degree leads to greater separation between  $P_\kappa$  and  $P_{-\kappa}$ . Interestingly, the variation accounted for by the proportion of pendants measure has been subsumed by the range in degree measure. The same result holds for the standard deviation of degree. Range or variance in degree is probably a better predictor of low values of  $\psi$  than the proportion of pendant nodes.

**Table 3**

	Model 1	Model 2	Model 3	Model 4	Model 5
Intercept	0.442 *** (0.019)	0.199 *** (0.025)	0.517 *** (0.017)	0.517 *** (0.016)	-0.019 (0.036)
Size	-0.016 *** (0.001)	-0.011 *** (0.001)	-0.018 *** (0.001)	-0.014 *** (0.001)	0.000 (0.001)
Prop. chain-link nodes	0.097 ** (0.03)				0.269 *** (0.03)
Density		0.475 *** (0.036)			0.740 *** (0.04)
Proportion of pendants			-0.270 *** (0.056)		0.092 † (0.054)
Range of node degree				-0.022 *** (0.004)	-0.030 *** (0.004)

N = 1,000, † p<0.1, \* p<0.05, \*\* p<0.01, \*\*\* p<0.001

Nevertheless, it is probably always worthwhile to check for a low value of  $\psi$  as an indication of the separation between the  $P_{\kappa}$  and  $P_{-\kappa}$ , rather than making a determination simply by looking at these structural properties alone.

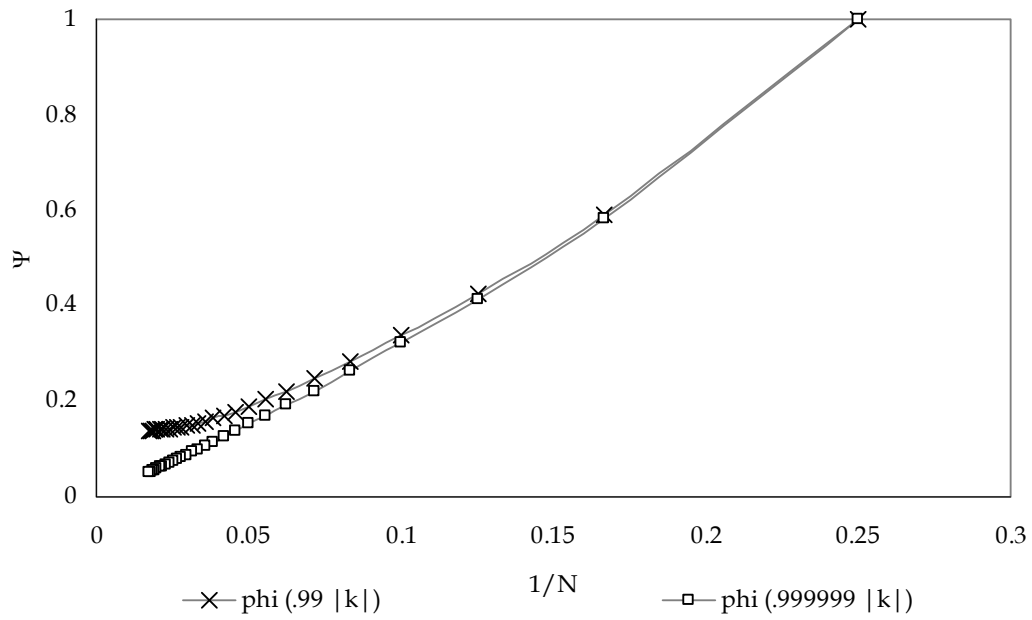
Finally, as to my comment about the meaning of  $\beta$  being “far from clear,” I should emphasize that this applies only to the cases in which intermediate values of  $\beta$  are not explained by a linear combination of  $-\kappa$  and  $+\kappa$  and in my simulations this was only a very small proportion. Generally, intermediate values  $\beta$  can be represented by  $P_{-\kappa}$  and  $P_{+\kappa}$  where intermediate  $\beta$  values might be interpreted as indicative of a ‘mixed strategy’ in which both positive and negative influences of association are present.

However, in cases in which  $\beta$  is not decomposable into the two components, interpretation becomes problematic. If one grants that  $P_{-\kappa}$  represents negative resource flows and  $P_{\kappa}$  positive flows, the inability to completely decompose an intermediate value into the two extremes suggest that it is capturing something that doesn’t lie on this single positive-negative flow dimension. It is this that makes interpretation difficult. The link between theory and measure works well for intermediate values as long as the power indices for the network are decomposable; in these cases, my suggestion is to use  $P_{-\kappa}$  and  $P_{\kappa}$  together. However, it is when they are not that the meaning of any findings based on intermediate values is hard to interpret.



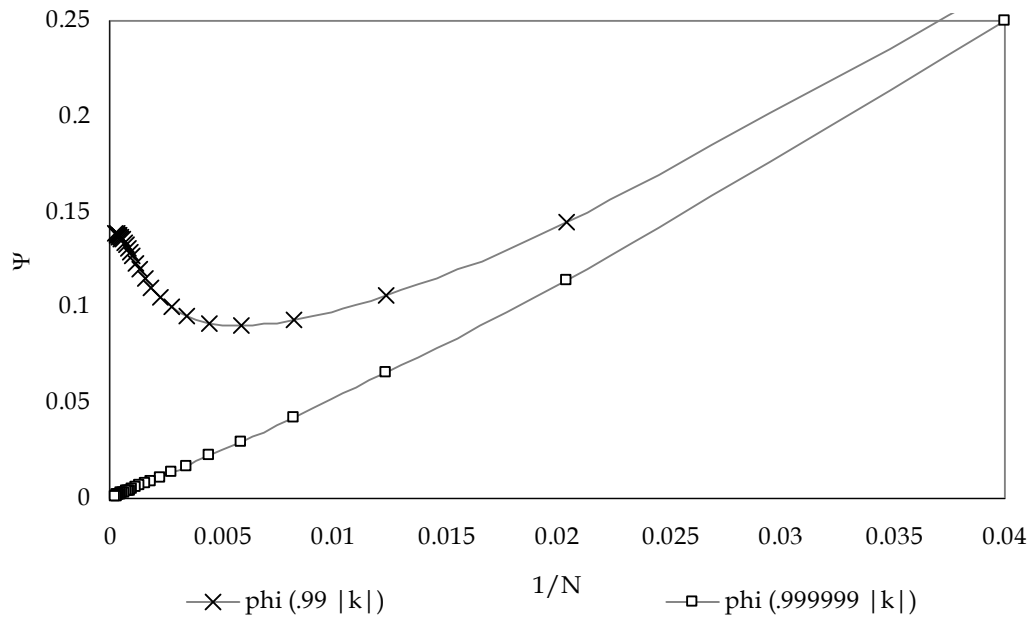
**Figure 1a**

*$\Psi$  plotted against  $1/N$  for even length chains*



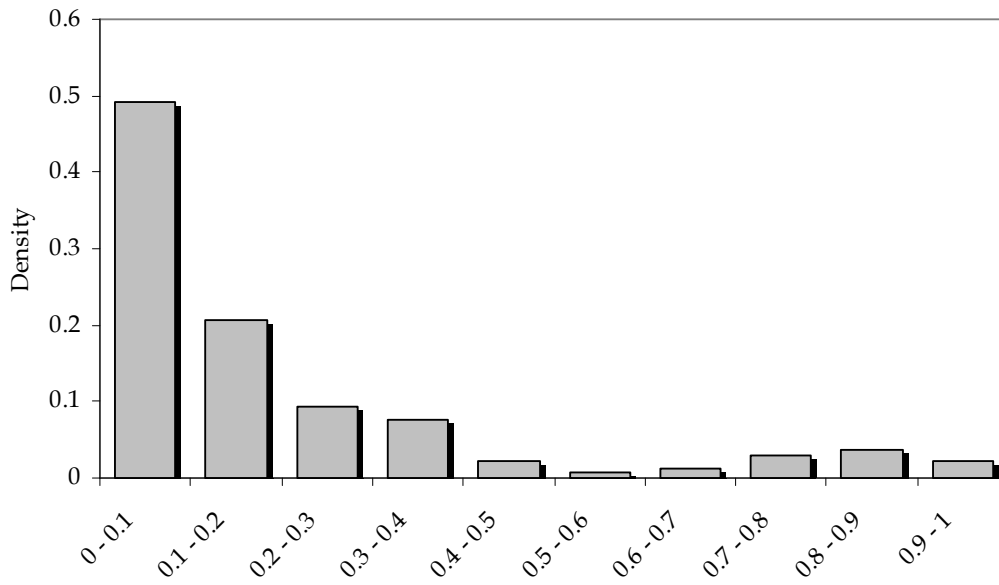
**Figure 1b**

*$\Psi$  plotted against  $1/N^2$  for odd length chains*



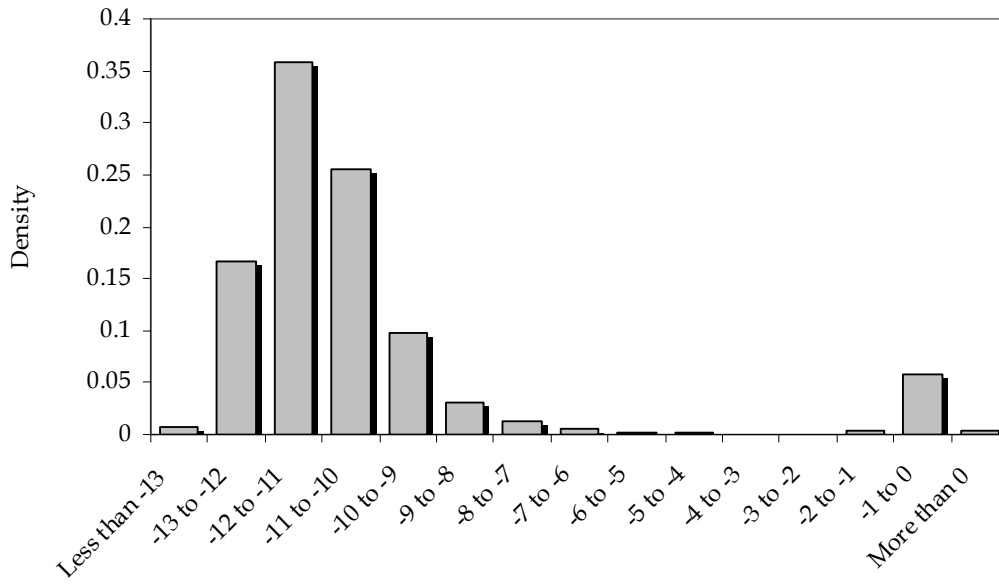
**Figure 2a**

*Frequency,  $\psi$*



**Figure 2b**

*Frequency,  $\ln(\gamma)$*



## References

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Rodan, S. (2011). "Choosing the  $\beta$  Parameter When Using the Bonacich Power Measure." *Journal of Social Structure* 11:4. Available: <http://www.cmu.edu/joss/content/articles/volume12/Rodan.pdf>