COWS and Their Hybrids

Customized Orthogonal Weights

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Joint work with: Mikael Kuusela and Chad Schafer

sWeights and COWS

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- This talk explains the method, our statistical deconstruction and some extensions (hybrids).

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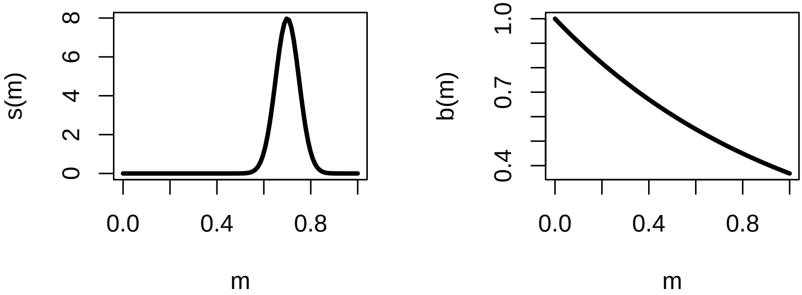
- We want to extract the signal component for T.

Signal and Background



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- The goal is to recover h_1 and h_2.

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- We return to this point later.

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- Under standard conditions, $||\widehat{h}_1 - h_1|| \xrightarrow{P} 0$ and $||\widehat{h}_2 - h_2|| \xrightarrow{P} 0$.

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where $B = A^{-1}$ and

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- The optimal weight functions depend on h_1 and h_2. (Can be done iteratively).

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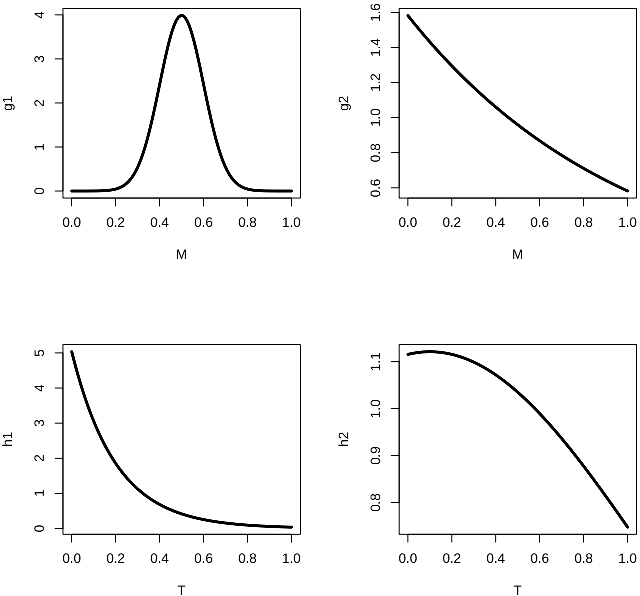
- But then

$$\mathbb{E}\left[\frac{\widetilde{w}(M) I(T = t)}{\pi}\right] = h_1(t) \int \frac{\widetilde{w}(m)}{P(S = 1|M = m, T = t)} g_1(m) dm \neq h_1(t).$$

Example



g1
0
1
2
3
4
0.0
0.2
0.4
0.6
0.8
1.0
M
g2
0.6
0.8
1.0
1.2
1.4
1.6
0.0
0.2
0.4
0.6
0.8
1.0
M
h1
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
T
h2
0.8
0.9
1.0
1.1
0.0
0.2
0.4
0.6
0.8
1.0
T

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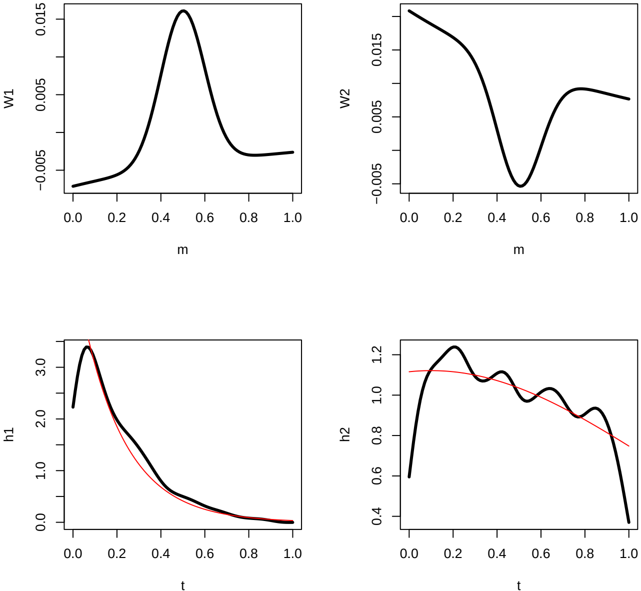
W1
0.015
0.005
−0.005
0.0
0.2
0.4
0.6
0.8
1.0
m
W2
0.015
0.005
−0.005
m
h1
3.0
2.0
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- Require that $\mathcal{G}_s$ and $\mathcal{G}_b$ are linearly independent.

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- How do we construct the herd?

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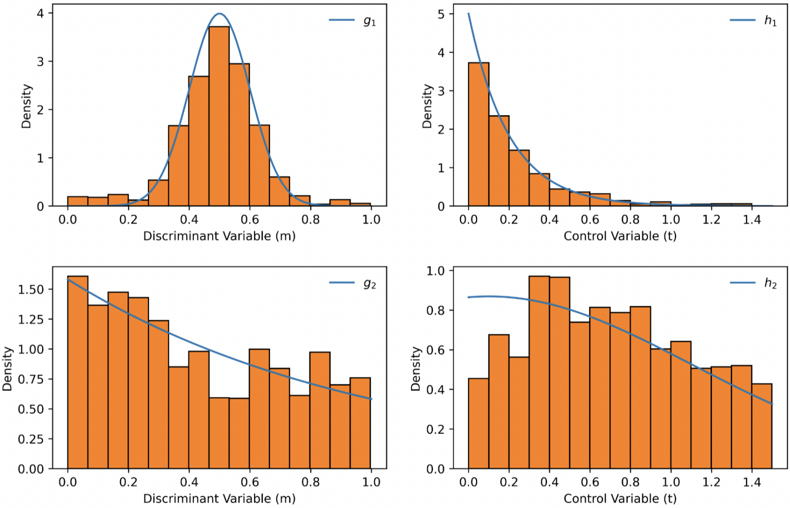

g_1
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Discriminant Variable (m)
h_1
Density
Control Variable (t)
g_2
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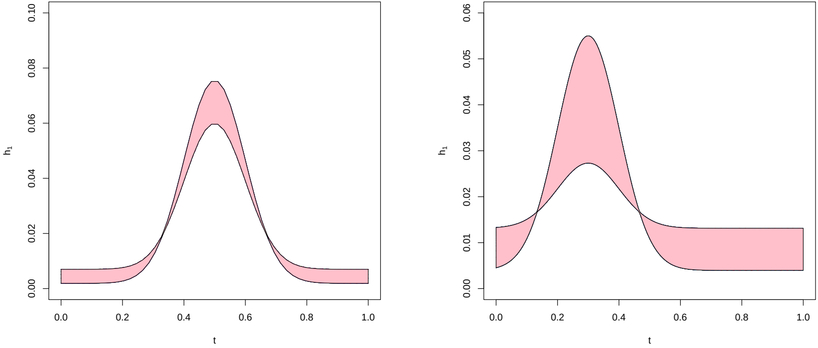
- From this, we can find bounds on $h_1(t)$.

Two Examples



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- COWs offers a different (easier) way to fit this model.

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- Then iterate:

$$w_{ij} = \frac{\pi_j \prod_{r=1}^d \mathcal{N}(f_{jr})(X_{ir})}{\sum_j \pi_j \prod_{r=1}^d \mathcal{N}(f_{jr})(X_{ir})}$$

$$\pi_j = \frac{1}{n} \sum_i w_{ij}$$

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- Two hybrids:
  - agnostic COWs
  - robust COWs

Agnostic COWs: No Marginal Model

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2	f_{21}	f_{22}	$\cdots$	f_{2d}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
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True densities are a fixed point.

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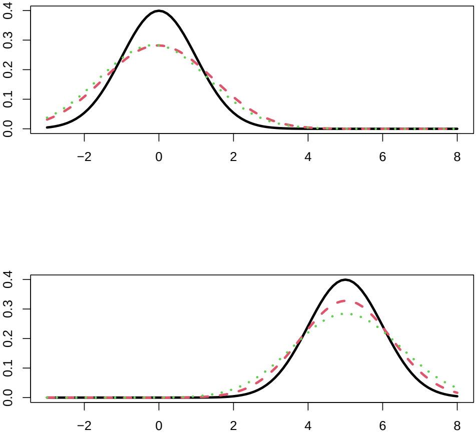
- Multiple robust: If at least one model is correct then $||\widehat{p}^{\widehat{r}} - p|| \xrightarrow{P} 0$ and

$$||\widehat{p}^{\widehat{r}} - p|| = O_P(n^{-1/(2\beta+1)}).$$

Example: Three Models, Third Misspecified



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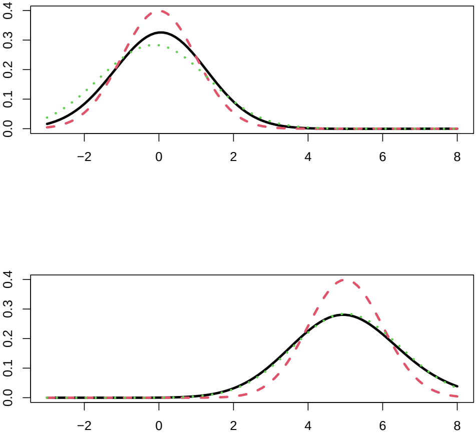

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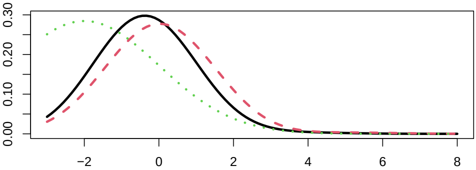

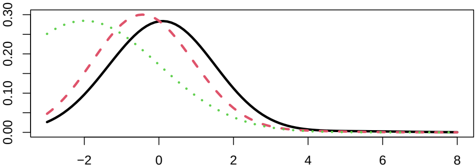

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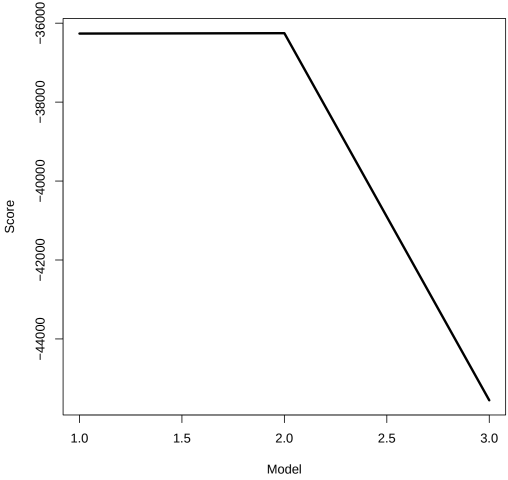

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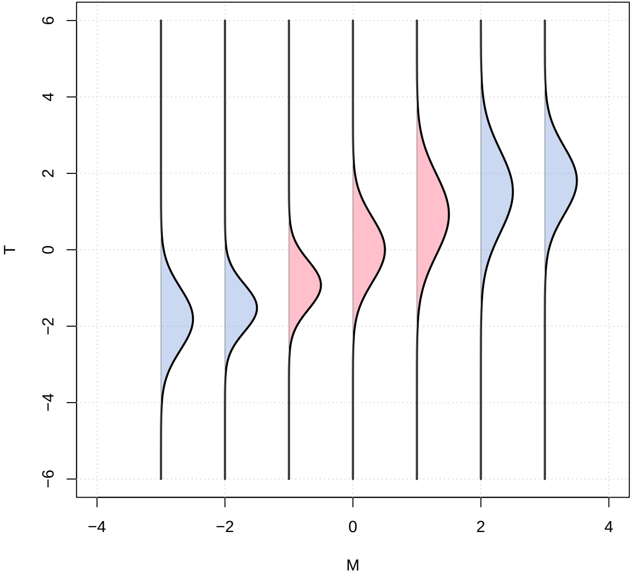

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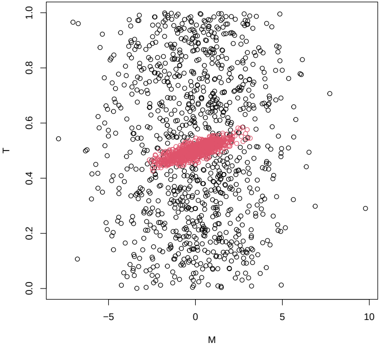
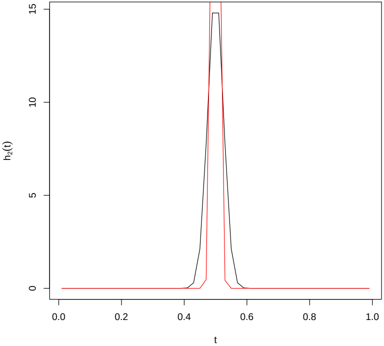
- EM is especially easy for the Gaussian copula.

Example: Overlapping Signals





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