

Literature Review of Optimal Transport and Riemannian Geometry



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Introduction

Based on 2 semesters of advanced undergraduate direct reading courses on Optimal Transport and Differential Geometry, I have culminated with a series of notes on respective theories as well as their connections. Main topics include the formulation of Monge Problem, Kantorovich Problem, Dual Problem, as well as a variational formulation of Dual Problem, and the Semi-Dual objective of optimal transport.

Dual Problem

We rewrite marginal-density constraint on a transference plan into a penalty term, then change minimax into maximin problem.

Semi-Dual Objective L(V,T) (Raymond Chu et al., 2026)

Equivalent condition for attaining arginf L(V,T):

Knothe-Rosenblatt Rearrangement

Consider 2 probability measures μ and ν over \mathbb{R}^d . Suppose $\forall A \in \mathcal{B}(\mathbb{R}^d)$, then $\mu(A) = \int_{\mathbb{R}^d} \mathbb{1}_A(x) d\mu(x)$. Where $\mathcal{B}(\mathbb{R}^d)$ is the Borel sigma-algebra of \mathbb{R}^d .
Also, assume $\mu \ll \nu$ (i.e. $\forall A \in \mathcal{B}(\mathbb{R}^d)$, if $\nu(A) = 0$, then $\mu(A) = 0$).
Define densities f^d and g^d as:
 $f^d(x) = \int_{\mathbb{R}^d} f(t_1, \dots, t_d) dt_1 \dots dt_d$
 $g^d(x) = \int_{\mathbb{R}^d} g(s_1, \dots, s_d) ds_1 \dots ds_d$
Also, for $k \leq d$, define $f^k(x_1, \dots, x_k) = \int_{\mathbb{R}^{d-k}} f(t_1, \dots, t_k, s_{k+1}, \dots, s_d) ds_{k+1} \dots ds_d$
Recall $\pi_k: \mathbb{R}^d \rightarrow \mathbb{R}^k$ (for some $A \in \mathcal{B}(\mathbb{R}^d)$)
Recall $(\pi_k)_\# \mu(A) = \mu(\pi_k^{-1}(A)) = \int_{\mathbb{R}^d} \mathbb{1}_A(\pi_k(x)) d\mu(x)$

By Tonelli Theorem, thus $\int_{\mathbb{R}^d} f(x) d\mu(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t_1, \dots, t_d) dt_1 \dots dt_d d\mu(x)$
only depend on x_d hence the outer-most layer of integration is w.r.t. x_d .
In fact, $\int_{\mathbb{R}^d} f(x) d\mu(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t_1, \dots, t_d) dt_1 \dots dt_d d\mu(x)$
only need to be non-negative and measurable to invoke Tonelli's.

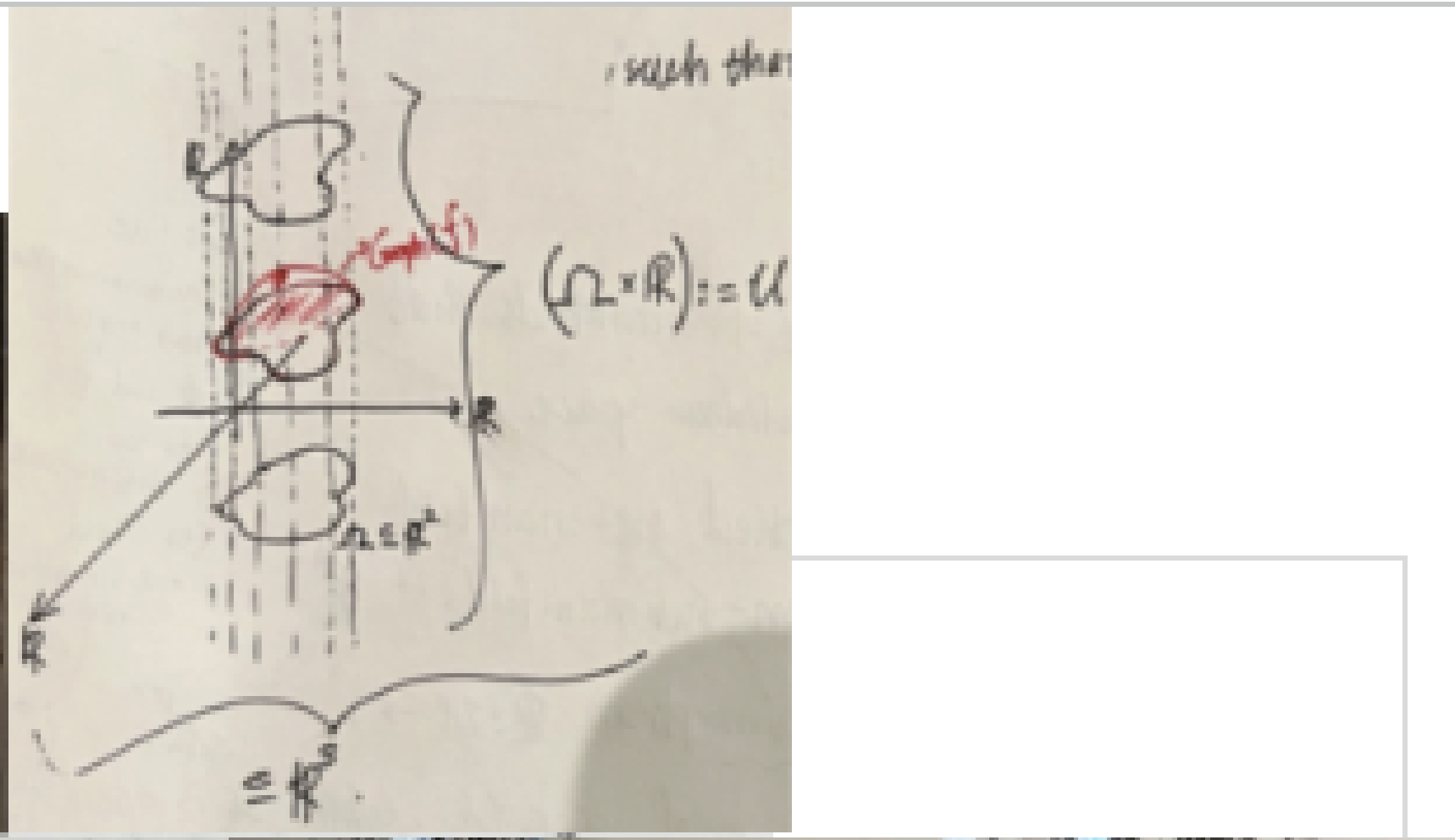
Also Raymond et al. have proved the tangential touching lemma.
Recall: Let $M \subseteq \mathbb{R}^d$ be a smooth, embedded Riemannian manifold.
Suppose $f \in C^1(\mathbb{R}^d)$ and $\varphi \in C^1(M)$, where f, φ share the same co-domain $\subseteq \mathbb{R}^l$.
such that $\varphi(x) \leq f(x), \forall x \in M$, and $\varphi(x^*) = f(x^*)$, for some $x^* \in M$.
Then $(\nabla_M \varphi(x^*)) = \Pi_{T_{x^*}M} \nabla f(x^*)$.
Remark: $(\Pi_{T_{x^*}M} \nabla f(x^*))$ is the orthogonal projection of $(\nabla f(x^*)) \in \mathbb{R}^d$ onto (vector subspace) $(T_{x^*}M) \subseteq \mathbb{R}^d$.
i.e., $\Pi_{T_{x^*}M} \nabla f(x^*)$ is the unique vector in $T_{x^*}M$, such that $\forall \vec{v} \in T_{x^*}M$, we have $\langle \vec{v}, (\nabla f(x^*)) - \Pi_{T_{x^*}M} \nabla f(x^*) \rangle = 0$.
Importantly, $\Pi_{T_{x^*}M} \nabla f(x^*) \in T_{x^*}M$.
Without loss of generality, let $\psi: U \rightarrow \psi(U)$ be a local chart, with $\psi(u) \ni x^*$.
To show: $\forall \vec{v} \in T_{x^*}M$, $\langle \vec{v}, \nabla_M \varphi(x^*) \rangle = \langle \vec{v}, \Pi_{T_{x^*}M} \nabla f(x^*) \rangle$.

Great Visual Intuition of Manifolds

Geometric Intuitions of Manifolds

Note that patches can intersect.
 $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$
 $\mathcal{U}_1 \cap \mathcal{U}_2 \subseteq S$

such that $(M \cap U) = \Phi^{-1}(R^m \times \{0\})$
i.e., $\exists \mathcal{C}(M \cap U) = \mathbb{R}^m \times \{0\}$
i.e. let U be an open neighborhood of x_0 .
 \mathbb{R}^m is m -dimensional.
 \mathbb{R}^n is n -dimensional.
 $\mathbb{R}^m \times \{0\}$ is m -dimensional.
think of \mathbb{R}^m as a plane in \mathbb{R}^n .



HW5. Q3. Let $S \subseteq \mathbb{R}^3$ be a smooth surface. For $p, q \in S$, we define the (in-surface) distance between p and q to be $d_S(p, q) = \inf \int_0^1 |\dot{\gamma}(t)| dt$ where $\gamma: [a, b] \rightarrow S$ is smooth and $\gamma(a) = p, \gamma(b) = q$.
arc-length along parametrization
of some curve connecting p with q .

Intuition: $\text{Im}(Df|_U) = T_{f(u)}M$, $\forall u \in U$, and (T_uM) is m -dimensional.
 $(T_uM) :=$ the set of velocities of curves in M through $f(u)$
 $= \{ \dot{\gamma}(0) : \gamma(t) \in M, \gamma(0) = f(u) \}$
claim: (move away from the perspective of $(Df|_U)$ as a linear transformer
Instead view $(Df|_U)$ as a linear transformer

Let $M \subseteq \mathbb{R}^n$ be a m -dimensional manifold.
Let $f: U \rightarrow \mathbb{R}^n$ be a local chart.
If $\alpha: [a, b] \rightarrow \mathbb{R}^n$ is a curve, then \exists curve $\beta: [a, b] \rightarrow M$ such that $f \circ \beta = \alpha$. (i.e. such that $\forall t \in [a, b]$ we have $\beta(t) \in M$ and $f(\beta(t)) = \alpha(t)$)
Since local chart $f: U \rightarrow \mathbb{R}^n$ is a homeomorphism, $f(\beta(t)) = \alpha(t)$
 $\beta = f^{-1} \circ \alpha$

Then (the immersion theorem) Let $f: \Omega \rightarrow \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^m$. Let $m < n$.
Suppose f is an immersion at some $x_0 \in \Omega$.
Then \exists some neighborhood U of x_0 such that $f(U) \subseteq \mathbb{R}^n$ and \exists diffeomorphism $\Phi: U \rightarrow \mathbb{R}^m \times \{0\}$ such that $f \circ \Phi^{-1} = \text{inclusion}$.
i.e., $f(U) \subseteq \mathbb{R}^m \times \{0\}$ and $f|_U$ is a diffeomorphism onto its image.
 \mathbb{R}^m (has an equivalent \mathbb{R}^m as a manifold (subset \mathbb{R}^n is))
representing the shaded area as $U \cap \mathbb{R}^m$ (where $U \ni x_0$) although $U \subseteq \mathbb{R}^m = (U \cap \mathbb{R}^m) = U$.
 \mathbb{R}^m (in fact, $\mathbb{R}^m \subseteq \mathbb{R}^n$)
 \mathbb{R}^m (in fact, $\mathbb{R}^m \subseteq \mathbb{R}^n$)

Theorem: $[Df|_U] = (D\Phi|_U) \circ [Df|_U] \circ (D\Phi|_U)^{-1}$, suppose $u \in U$.
 $(Df|_U)_u = (D\Phi|_U)_u \circ [Df|_U]_u \circ (D\Phi|_U)_u^{-1}$
In order for this, this's direction to check, we must have $m_1 = m_2$ and $m_1 = m_2$.
Suppose M is a Riemannian manifold with the metric $\langle \cdot, \cdot \rangle_M = \langle \cdot, \cdot \rangle_{\mathbb{R}^n} \circ Df|_U$.
Recall each g_{ij} coefficient corresponds to a parametrization/local chart $f: U \rightarrow M$.
Denote \tilde{g}_{ij} and \hat{g}_{ij} to be the coefficients of Riemannian metric on M corresponding respectively to local charts f and \tilde{f} .
 $\tilde{g}_{ij} = \langle \tilde{f}_i, \tilde{f}_j \rangle_M = \langle D\tilde{f}|_U(\tilde{f}_i), D\tilde{f}|_U(\tilde{f}_j) \rangle_{\mathbb{R}^n}$
 $\hat{g}_{ij} = \langle f_i, f_j \rangle_M = \langle Df|_U(f_i), Df|_U(f_j) \rangle_{\mathbb{R}^n}$
(thus free M)
 $\hat{g}_{ij} = \langle Df|_U(f_i), Df|_U(f_j) \rangle_{\mathbb{R}^n}$